EXOTIC BAILEY-SLATER SPT-FUNCTIONS III: BAILEY PAIRS FROM GROUPS B, F, G, AND J

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Abstract. We continue to investigate spt-type functions that arise from Bailey pairs. In this third paper on the subject, we proceed to introduce additional spt-type functions. We prove simple Ramanujan type congruences for these functions which can be explained by a spt-crank-type function. The spt-crank-type functions are actually defined first, with the spt-type functions coming from setting $z = 1$ in the spt-crank-type functions. We find some of the spt-crank-type functions to have interesting representations as single series, some of which reduce to infinite products. Additionally we find dissections of the other spt-crank-type functions when z is a certain root of unity. Both methods are used to explain congruences for the spt-type functions. Our series formulas require Bailey's Lemma and conjugate Bailey pairs. Our dissection formulas follow from Bailey's Lemma and dissections of known ranks and cranks.

1. Introduction

We proceed with the study of spt-crank-type functions that the author began in [18] and continued in [15]. We begin with a brief introduction. We recall a partition of n is a non-increasing sequence of positive integers that sum to n. For example, the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. We have Andrews smallest parts function from [3], $\operatorname{spt}(n)$, as the weighted count on partitions given by counting a partition by the number of times the smallest part appears. From the partitions of 4 we see that $\operatorname{spt}(4) = 10$. In this article we consider variations of the smallest parts functions. We use the standard product notation

$$
\begin{aligned}
(z;q)_n &= \prod_{j=0}^{n-1}(1-zq^j), & (z;q)_\infty &= \prod_{j=0}^{\infty}(1-zq^j),\\
(z_1,\dots,z_k;q)_n &= (z_1;q)_n\dots(z_k;q)_n, & (z_1,\dots,z_k;q)_\infty &= (z_1;q)_\infty\dots(z_k;q)_\infty,\\
[z;q]_\infty &= (z,q/z;q)_\infty, & [z_1,\dots,z_k;q]_\infty &= [z_1;q]_\infty\dots[z_k;q]_\infty.
\end{aligned}
$$

We recall that a pair of sequences (α,β) is a Bailey pair relative to (a,q) if

$$\beta_n = \sum_{k=0}^{n}\frac{\alpha_k}{(q;q)_{n-k}\,(aq;q)_{n+k}}.$$

One may consult [2] for a history of Bailey pairs and Bailey's Lemma. Motivated by the prototype spt-crank functions of [5] and [14] for partitions and overpartitions, we consider an spt-crank-type function to be a function of the form

$$\frac{P(q)}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\left(z,z^{-1};q\right)_n q^n\beta_n,$$

where $P(q)$ is some product and β comes from a Bailey pair relative to $(1,q)$. We consider an spt-type function to be the $z=1$ case of an spt-crank-type function. That is, for a Bailey pair (α^X,β^X) relative to $(1,q)$ we have the spt-crank-type and spt-type functions given by

$$S_X(z,q) = \frac{P_X(q)}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\left(z,z^{-1};q\right)_n q^n\beta_n^X = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}M_X(m,n)z^mq^n,$$

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$$S_X(q) = S_X(1,q) = \sum_{n=1}^{\infty} \operatorname{spt}_X(n)\, q^n.$$

The author is in the process of studying interesting spt-crank-type and spt-type functions. This article introduces the last of the spt-crank-type and spt-type functions arising from Bailey pairs in [26] and [27] that possess simple linear congruences of the form $\operatorname{spt}_X(pn+b) \equiv 0 \pmod{p}$, where p is an odd prime. In [18] the author introduced the spt-crank-type functions $S_{A1}(z,q)$, $S_{A3}(z,q)$, $S_{A5}(z,q)$, and $S_{A7}(z,q)$ which correspond to the Bailey pairs $A(1)$, $A(3)$, $A(5)$, and $A(7)$ of [26]. In [15] Garvan and the author introduced the spt-crank-type functions $S_{C1}(z,q)$, $S_{C5}(z,q)$, $S_{E2}(z,q)$, and $S_{E4}(z,q)$. These spt-type functions satisfy many linear congruences, in particular, $\operatorname{spt}_{A1}(3n) \equiv \operatorname{spt}_{A3}(3n+1) \equiv \operatorname{spt}_{E2}(3n) \equiv \operatorname{spt}_{E4}(3n+1) \equiv 0 \pmod{3}$, $\operatorname{spt}_{A3}(5n+1) \equiv \operatorname{spt}_{A5}(5n+4) \equiv \operatorname{spt}_{A7}(5n+1) \equiv \operatorname{spt}_{C1}(5n+3) \equiv \operatorname{spt}_{C5}(5n+3) \equiv 0 \pmod{5}$, and $\operatorname{spt}_{A5}(7n+1) \equiv 0 \pmod{7}$. Here we consider the Bailey pairs $B(2)$, $F(3)$, $G(4)$, and the entry just above $G(4)$ from [26] and $J(1)$, $J(2)$, and $J(3)$ from [27].

We prove simple linear congruences for the $\operatorname{spt}_X(n)$ by considering $S_X(\zeta,q)$, where ζ is a root of unity. For t a positive integer we define

$$M_X(k,t,n) = \sum_{m \equiv k \pmod{t}} M_X(m,n).$$

We note that

$$\operatorname{spt}_X(n) = \sum_{k=0}^{t-1} M_X(k,t,n).$$

When ζ_t is a t^{th} root of unity, we have

$$S_X(\zeta_t,q) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{t-1} M_X(k,t,n)\zeta_t^k \right) q^n.$$

The last equation is of great importance because if t is prime and ζ_t is a primitive t^{th} root of unity, then the minimal polynomial for ζ_t is $1 + x + x^2 + \cdots + x^{t-1}$. Thus if the coefficient of q^N in $S_X(\zeta_t,q)$ is zero, then $\sum_{k=0}^{t-1} M_X(k,t,N)\zeta_t^k$ is zero and so $M_X(0,t,N) = M_X(1,t,N) = \cdots = M_X(t-1,t,N)$. But then we would have that $\operatorname{spt}_X(N) = t \cdot M_X(0,t,N)$ and so if $M_X(0,t,N)$ is an integer then clearly $\operatorname{spt}_X(N) \equiv 0 \pmod{t}$. That is to say, if the coefficient of q^N in $S_X(\zeta_t,q)$ is zero, then $\operatorname{spt}_X(N) \equiv 0 \pmod{t}$. Thus not only do we have the congruence $\operatorname{spt}_X(N) \equiv 0 \pmod{t}$, but also the stronger combinatorial result that all of the $M_X(r,t,N)$ are equal.

In [18] the author found dissection formulas for the $S_{Ai}(z,q)$ when z is the appropriate root of unity to establish the various congruences. In [15] Garvan and the author similarly found dissection formulas for the $S_{Ci}(z,q)$ and $S_{Ei}(z,q)$ when z is the appropriate root of unity. The main difference between these two papers is that the $S_{Ci}(z,q)$ and $S_{Ei}(z,q)$ could be expressed in terms of functions with known dissections, whereas the $S_{Ai}(z,q)$ could not. Additionally in [15], we found interesting series representations for the $S_{Ci}(z,q)$, $S_{Ei}(z,q)$, and the $S_{Ai}(z,q)$ that are valid for all values of z, rather than just a fixed root of unity. These series representations were a combination of single series representations that showed that some of the spt-crank-type functions could be written just in terms of infinite products, and double series representations that could be written as so called Hecke-Rogers type double sums.

In the next section we define the new spt-crank-type and spt-type functions and state our main results, which are congruences for the various spt-type functions, single series representations for some of the spt-crank-type functions, and dissection formulas for the other spt-crank-type functions.

2. Preliminaries and Statement of Results

To begin we use the Bailey pair $B(2)$ from [26] and $J(1)$, $J(2)$, and $J(3)$ from [27]. Each of these is a Bailey pair relative to $(1,q)$ and in all cases $\alpha_0 = \beta_0 = 1$.

$$\beta_n^{B2} = \frac{q^n}{(q;q)_n}, \qquad \alpha_n^{B2} = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{3(n^2-n)/2}(1+q^{3n}) & \text{if } n \geq 1 \end{cases},$$

$$\beta_n^{J1} = \frac{\left(q^3;q^3\right)_{n-1}}{(q;q)_{2n-1}\,(q;q)_n}, \qquad \alpha_n^{J1} = \begin{cases} 0 & \text{if } n=3k-1 \\ (-1)^k q^{\frac{9k^2-3k}{2}}(1+q^{3k}) & \text{if } n=3k \\ 0 & \text{if } n=3k+1 \end{cases},$$

$$\beta_n^{J2} = \frac{\left(q^3;q^3\right)_{n-1}}{(q;q)_{2n}\,(q;q)_{n-1}}, \qquad \alpha_n^{J2} = \begin{cases} (-1)^{k-1} q^{\frac{9k^2-9k+2}{2}} & \text{if } n=3k-1 \\ (-1)^k q^{\frac{9k^2-3k}{2}}(1+q^{3k}) & \text{if } n=3k \\ (-1)^{k+1} q^{\frac{9k^2+9k+2}{2}} & \text{if } n=3k+1 \end{cases},$$

$$\beta_n^{J3} = \frac{q^n\left(q^3;q^3\right)_{n-1}}{(q;q)_{2n}\,(q;q)_{n-1}}, \qquad \alpha_n^{J3} = \begin{cases} (-1)^{k-1} q^{\frac{9k^2-3k}{2}} & \text{if } n=3k-1 \\ (-1)^k q^{\frac{9k^2-3k}{2}}(1+q^{3k}) & \text{if } n=3k \\ (-1)^{k+1} q^{\frac{9k^2+3k}{2}} & \text{if } n=3k+1 \end{cases}.$$

We note these Bailey pairs from group J also appear as unlabeled Bailey pairs on page 467 of [26]. Additionally, we use the following Bailey pairs relative to $(1,q^2)$, from [26]:

$$\beta_n^{F3} = \frac{q^{-n}}{(q,q^2;q^2)_n}, \qquad \alpha_n^{F3} = \begin{cases} 1 & \text{if } n=0 \\ q^n+q^{-n} & \text{if } n\ge 1 \end{cases},$$

$$\beta_n^{G4} = \frac{(-1)^n q^{n^2}}{(q^4;q^4)_n\,(-q;q^2)_n}, \qquad \alpha_n^{G4} = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(n-1)/2}(1+q^n) & \text{if } n\ge 1 \end{cases},$$

$$\beta_n^{AG4} = \frac{(-1)^n q^{n^2-2n}}{(q^4;q^4)_n\,(-q;q^2)_n}, \qquad \alpha_n^{AG4} = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(n-3)/2}(1+q^{3n}) & \text{if } n\ge 1 \end{cases}.$$

The Bailey pair $AG(4)$ is the entry just above $G(4)$ in [26]. For each Bailey pair we define a two variable spt-crank-type series as follows,

$$S_{B2}(z,q) = \frac{(q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q\right)_n q^{2n}}{(q;q)_n},$$

$$S_{F3}(z,q) = \frac{(q;q)_\infty}{(z,z^{-1};q^2)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n q^{n}}{(q;q)_{2n}},$$

$$S_{G4}(z,q) = \frac{\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty}{(z,z^{-1};q^2)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n (-1)^n q^{n^2+2n}}{\left(q^4;q^4\right)_n\left(-q;q^2\right)_n},$$

$$S_{AG4}(z,q) = \frac{\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty}{(z,z^{-1};q^2)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n (-1)^n q^{n^2}}{\left(q^4;q^4\right)_n\left(-q;q^2\right)_n},$$

$$S_{J1}(z,q) = \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q\right)_n\left(q^3;q^3\right)_{n-1} q^{n}}{(q;q)_{2n-1}\,(q;q)_n},$$

$$S_{J2}(z,q) = \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q\right)_n\left(q^3;q^3\right)_{n-1} q^{n}}{(q;q)_{2n}\,(q;q)_{n-1}},$$

$$S_{J3}(z,q) = \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q\right)_n\left(q^3;q^3\right)_{n-1} q^{2n}}{(q;q)_{2n}\,(q;q)_{n-1}}.$$

While it is not true that $J(1)=J(2)+J(3)$, because $\beta_0^{J1}=1$, we do have $\beta_n^{J1}=\beta_n^{J2}+\beta_n^{J3}$ for $n\ge 1$ and so $S_{J1}(z,q)=S_{J2}(z,q)+S_{J3}(z,q)$.

Next we define the corresponding spt-type functions. For $B2$, $J1$, $J2$, and $J3$ we just set $z=1$ and simplify the products, but for $F3$, $G4$, and $AG4$ we make some additional rearrangements.

$$S_{B2}(q) = \sum_{n=1}^{\infty}\mathrm{spt}_{B2}(n)\,q^n = \sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^n)^2\,(q^{n+1};q)_\infty},$$

$$
\begin{aligned}
S_{J1}(q) &= \sum_{n=1}^{\infty} \operatorname{spt}_{J1}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 \left(q^{n+1};q\right)_{n-1} \left(q^{3n};q^3\right)_\infty},\\
S_{J2}(q) &= \sum_{n=1}^{\infty} \operatorname{spt}_{J2}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n}{\left(q^{n};q\right)_{n+1} \left(q^{3n};q^3\right)_\infty},\\
S_{J3}(q) &= \sum_{n=1}^{\infty} \operatorname{spt}_{J3}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{2n}}{\left(q^{n};q\right)_{n+1} \left(q^{3n};q^3\right)_\infty},\\
S_{F3}(q) &= \sum_{n=1}^{\infty} \operatorname{spt}_{F3}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n \left(q^{2n+1};q^2\right)_\infty}{(1-q^{2n})^2 \left(q^{2n+2};q^2\right)_\infty}\\
&= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^{2n})^2 \left(q^{n+1};q\right)_{n} \left(q^{2n+2};q^2\right)_\infty} \frac{\left(q^{n+1};q\right)_\infty}{\left(q^{2n+2};q^2\right)_\infty},\\
S_{G4}(q) &= \sum_{n=1}^{\infty} \operatorname{spt}_{G4}(n)\, q^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+2n} \left(-q^{2n+1};q^2\right)_\infty}{(1-q^{2n})^2 \left(q^{2n+2};q^2\right)_{n+1} \left(q^{2n+2};q^2\right)_\infty \left(q^{4n+6};q^4\right)_\infty}\\
&= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1-q^{2n})^2 \left(q^{n+1};q\right)_{n} \left(q^{2n+2};q^2\right)_{n+1} \left(q^{4n+6};q^4\right)_\infty} \frac{\left(q^{n+1};q\right)_n \left(-q^{2n+1};q^2\right)_\infty}{\left(q^{2n+2};q^2\right)_\infty},\\
S_{AG4}(q) &= \sum_{n=1}^{\infty} \operatorname{spt}_{G4}(n)\, q^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2} \left(-q^{2n+1};q^2\right)_\infty}{(1-q^{2n})^2 \left(q^{2n+2};q^2\right)_{n+1} \left(q^{2n+2};q^2\right)_\infty \left(q^{4n+6};q^4\right)_\infty}\\
&= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(1-q^{2n})^2 \left(q^{n+1};q\right)_{n} \left(q^{2n+2};q^2\right)_{n+1} \left(q^{4n+6};q^4\right)_\infty} \frac{\left(q^{n+1};q\right)_n \left(-q^{2n+1};q^2\right)_\infty}{\left(q^{2n+2};q^2\right)_\infty}.
\end{aligned}
$$

We recall that an overpartition is a partition in which a part may be overlined the first time it appears; overpartitions can be identified with partition pairs (π_1, π_2) where π_2 is restricted to having distinct parts. For π either a partition or an overpartition, we let $s(\pi)$ denote the smallest part of π, $spt(\pi)$ denote the number of times $s(\pi)$ occurs, and $\#(\pi)$ denote the number of parts of π. For overpartitions we let a superscript n in these operators mean the restriction to the non-overlined parts and a superscript o mean the restriction to the overlined parts. For example, $\#^o(\pi)$ is the number of overlined parts of the overpartition π and $s^n(\pi)$ is the smallest non-overlined part. We can now give the combinatorial interpretation of the various spt-type functions.

We see $\operatorname{spt}_{B2}(n)$ is the number of partitions π of n weighted by the number of times $s(\pi)$ appears past the first occurrence. From this interpretation we see that $\operatorname{spt}_{B2}(n) = \operatorname{spt}(n) - p(n)$. We see $\operatorname{spt}_{J1}(n)$ is the number of partitions π of n weighted by the number of times $s(\pi)$ appears, where the allowed parts are those from $s(\pi)$ to $2s(\pi)-1$ and those that are divisible by 3 and at least $3s(\pi)$. We see $\operatorname{spt}_{J2}(n)$ is the number of partitions π of n where the parts are those from $s(\pi)$ to $2s(\pi)$ and those that are divisible by 3 and at least $3s(\pi)$. Similarly $\operatorname{spt}_{J3}(n)$ is the number of partitions π of n where the smallest part appears at least twice and the parts are those from $s(\pi)$ to $2s(\pi)$ and those that are divisible by 3 and at least $3s(\pi)$.

From the generating functions we see that $\operatorname{spt}_{J1}(n) = \operatorname{spt}_{J2}(n) + \operatorname{spt}_{J3}(n)$, as pointed out by the referee is it also not difficult to explain this combinatorially. Suppose we fix n and let $Ji(n)$ denote the set of partitions counted by $\operatorname{spt}_{Ji}(n)$. We have $J1(n)$ is the set of partitions π of n where no part π_i satisfies $2s(\pi) \le \pi_i < 3s(\pi)$, and if a part $\pi_i \ge 3s(\pi)$ then 3 divides π_i. Similarly $J2(n)$ is the set of partitions π of n where no part π_i satisfies $2s(\pi) < \pi_i < 3s(\pi)$, and if a part $\pi_i \ge 3s(\pi)$ then 3 divides π_i. Lastly $J3(n)$ is the set of partitions π of n where $s(\pi)$ appears at least twice, no part π_i satisfies $2s(\pi) \le \pi_i < 3s(\pi)$, and if a part $\pi_i \ge 3s(\pi)$ then 3 divides π_i. Given a partition $\pi \in J2(n)$, we obtain a partition of $J1(n)$ by taking the part $2s(\pi)$ and writing it as $s(\pi)+s(\pi)$, so the smallest part appears two more times for each time $2s(\pi)$ appeared; this function is clearly onto and is exactly $\left\lfloor \frac{spt(\pi)+1}{2} \right\rfloor$-to-one. Given a partition $\pi \in J3(n)$, we obtain a partition of $J1(n)$ in the same way, but now we miss the elements of $J1(n)$ whose smallest part appears exactly once and this function is $\left\lfloor \frac{spt(\pi)}{2} \right\rfloor$-to-one. Since $\left\lfloor \frac{spt(\pi)+1}{2} \right\rfloor + \left\lfloor \frac{spt(\pi)}{2} \right\rfloor = spt(\pi)$, we see $\operatorname{spt}_{J1}(n) = \operatorname{spt}_{J2}(n) + \operatorname{spt}_{J3}(n)$

For $S_{F3}(q)$, we first note that

$$\frac{q^n}{(1-q^{2n})^2} = q^n + 2q^{3n} + 3q^{5n} + 4q^{7n} + 5q^{9n} + \dots.$$

We let $F3$ denote the set of pairs (π_1, π_2), where π_1 is a partition with $spt(\pi_1)$ odd and the parts that are at least $2s(\pi)$ must even; and π_2 is an overpartition where all non-overlined parts are even, $s^n(\pi_2) \geq 2s(\pi_1)+2$, and $s^o(\pi_2) \geq s(\pi_1)+1$. Then we see that $\mathrm{spt}_{F3}(n)$ is the number of partition pairs (π_1, π_2) of n from $F3$, weighted by $(-1)^{\#^o(\pi_2)} \left(\frac{spt(\pi_1)+1}{2}\right)$.

For $S_{G4}(q)$, we first note that

$$\frac{q^{n^2+2n}}{(1-q^{2n})^2} = q^{n(n+2)} + 2q^{n(n+4)} + 3q^{n(n+6)} + 4q^{n(n+8)} + \dots.$$

We let $G4$ be the set of pairs (π_1, π_2) where π_1 is a partition such that $spt(\pi_1) \geq s(\pi_1)+2$, $spt(\pi_1)+s(\pi_1)$ is even, parts larger than $2s(\pi_1)$ must be even, and parts larger than $4s(\pi_1)$ must be congruent to $2 \pmod 4$; and π_2 is an overpartition with all non-overlined parts even, $s^n(\pi_2) \geq 2s(\pi_1)+2$, $s^o(\pi_2) \geq s(\pi)+1$, and overlined parts that are at least $2s(\pi_1)+1$ are odd. For an overpartition π, we let $k_m(\pi)$ denote the number of overlined parts of π that less than $2m+1$. Then $\mathrm{spt}_{G4}(n)$ is the number of partition pairs of n from $G4$, weighted by $(-1)^{s(\pi_1)+k_{s(\pi_1)}(\pi_2)} \left(\frac{spt(\pi_1)-s(\pi)}{2}\right)$.

For $S_{AG4}(q)$, we first note that

$$\frac{q^{n^2}}{(1-q^{2n})^2} = q^{n(n)} + 2q^{n(n+2)} + 3q^{n(n+4)} + 4q^{n(n+6)} + \dots.$$

We let $AG4$ be the set of pairs (π_1, π_2) where π_1 is a partition such that $spt(\pi_1) \geq s(\pi_1)$, $spt(\pi_1)+s(\pi_1)$ is even, parts larger than $2s(\pi_1)$ must be even, and parts larger than $4s(\pi_1)$ must be congruent to $2 \pmod 4$; and π_2 is an overpartition with all non-overlined parts even, $s^n(\pi_2) \geq 2s(\pi_1)+2$, $s^o(\pi_2) \geq s(\pi)+1$, and overlined parts that are at least $2s(\pi_1)+1$ are odd. Then $\mathrm{spt}_{AG4}(n)$ is the number of partition pairs of n from $AG4$, weighted by $(-1)^{s(\pi_1)+k_{s(\pi_1)}(\pi_2)} \left(\frac{spt(\pi_1)-s(\pi)+2}{2}\right)$.

These functions satisfy the following congruences.

Theorem 2.1.

$$\begin{aligned}
\mathrm{spt}_{F3}(3n) &\equiv 0 \pmod 3, \\
\mathrm{spt}_{J1}(3n+2) &\equiv 0 \pmod 3, \\
\mathrm{spt}_{J2}(3n) &\equiv 0 \pmod 3, \\
\mathrm{spt}_{J3}(3n+1) &\equiv 0 \pmod 3, \\
\mathrm{spt}_{B2}(5n+1) &\equiv 0 \pmod 5, \\
\mathrm{spt}_{B2}(5n+4) &\equiv 0 \pmod 5, \\
\mathrm{spt}_{F3}(5n) &\equiv 0 \pmod 5, \\
\mathrm{spt}_{F3}(5n+4) &\equiv 0 \pmod 5, \\
\mathrm{spt}_{G4}(5n+4) &\equiv 0 \pmod 5, \\
\mathrm{spt}_{AG4}(5n+4) &\equiv 0 \pmod 5, \\
\mathrm{spt}_{B2}(7n+1) &\equiv 0 \pmod 7, \\
\mathrm{spt}_{B2}(7n+5) &\equiv 0 \pmod 7, \\
\mathrm{spt}_{F3}(7n) &\equiv 0 \pmod 7, \\
\mathrm{spt}_{F3}(7n+4) &\equiv 0 \pmod 7, \\
\mathrm{spt}_{F3}(7n+6) &\equiv 0 \pmod 7.
\end{aligned}$$

That $\mathrm{spt}_{J1}(3n+2) \equiv 0$ is actually known. In [23] Patkowski considered this smallest parts function and proved that $\mathrm{spt}_{J1}(3n+2) \equiv 0$. Although that proof is dependent on Bailey's Lemma, the proof is not through a spt-crank-type function as we have here. Since $\mathrm{spt}_{B2}(n) = \mathrm{spt}(n) - p(n)$, the congruences

$\mathrm{spt}_{B2}(5n+4) \equiv 0 \pmod 5$ and $\mathrm{spt}_{B2}(7n+5) \equiv 0 \pmod 7$ also follow from the fact that both $\mathrm{spt}(n)$ and $p(n)$ satisfy these congruences. We use the spt-crank-type functions to prove the congruences of Theorem 2.1 as explained in the introduction. This will be as a corollary to the following two theorems.

Theorem 2.2.

$$\begin{aligned}&(1+z)\left(z,z^{-1},q;q\right)_\infty S_{J1}(z,q)\\&=\sum_{j=2}^{\infty}\frac{(1-z^{j-1})(1-z^j)z^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}(1-q^j-q^{2j-2}+q^{4j-3}+q^{5j-2}-q^{6j-3})}{(1-q^{3j-3})(1-q^{3j})},\end{aligned} \tag{2.1}$$

$$\begin{aligned}&(1+z)\left(z,z^{-1},q;q\right)_\infty S_{J2}(z,q)\\&=\sum_{j=2}^{\infty}\frac{(1-z^{j-1})(1-z^j)z^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}(1-q^{j-1}-q^{2j}+q^{4j-1}+q^{5j-3}-q^{6j-3})}{(1-q^{3j-3})(1-q^{3j})},\end{aligned} \tag{2.2}$$

$$\begin{aligned}&(1+z)\left(z,z^{-1},q;q\right)_\infty S_{J3}(z,q)\\&=\sum_{j=2}^{\infty}\frac{(1-z^{j-1})(1-z^j)z^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}(q^{j-1}-q^j-q^{2j-2}+q^{2j}+q^{4j-3}-q^{4j-1}-q^{5j-3}+q^{5j-2})}{(1-q^{3j-3})(1-q^{3j})},\end{aligned} \tag{2.3}$$

$$(1+z)\left(z,z^{-1},q;q^2\right)_\infty S_{F3}(z,q)=\sum_{j=-\infty}^{\infty}(1-z^{j-1})(1-z^j)z^{1-j}(-1)^{j+1}q^{(j-1)^2}, \tag{2.4}$$

$$(1+z)\left(z,z^{-1};q^2\right)_\infty S_{G4}(z,q)=\sum_{j=-\infty}^{\infty}(1-z^{j-1})(1-z^j)z^{1-j}q^{2j^2-j}, \tag{2.5}$$

$$(1+z)\left(z,z^{-1};q^2\right)_\infty S_{AG4}(z,q)=\sum_{j=-\infty}^{\infty}(1-z^{j-1})(1-z^j)z^{1-j}q^{2j^2+j}. \tag{2.6}$$

Theorem 2.3.

$$\begin{aligned}&S_{B2}(\zeta_5,q)\\&=1-\frac{\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{25}\right]_\infty}{\left[q^5;q^{25}\right]_\infty^2}+(1-\zeta_5-\zeta_5^4)q^5\frac{1}{\left(q^{25};q^{25}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2}}{1-q^{25n+5}}+q^2\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{25}\right]_\infty}\\&\quad+(\zeta_5+\zeta_5^4)q^3\frac{\left(q^{25};q^{25}\right)_\infty\left[q^5;q^{25}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty^2}+(\zeta_5+\zeta_5^4)q^8\frac{1}{\left(q^{25};q^{25}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2}}{1-q^{25n+10}},\end{aligned} \tag{2.7}$$

$$\begin{aligned}&S_{B2}(\zeta_7,q)\\&=\zeta_7+\zeta_7^6-(\zeta_7+\zeta_7^6)\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{21};q^{49}\right]_\infty}{\left[q^7,q^{14};q^{49}\right]_\infty}+(-1+\zeta_7+\zeta_7^6)q^7\frac{1}{\left(q^{49};q^{49}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{147n(n+1)/2}}{1-q^{49n+7}}\\&\quad+q^2\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{14};q^{49}\right]_\infty}{\left[q^7,q^{21};q^{49}\right]_\infty}+(1+\zeta_7^2+\zeta_7^5)q^{16}\frac{1}{\left(q^{49};q^{49}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{147n(n+1)/2}}{1-q^{49n+21}}\\&\quad+(\zeta_7+\zeta_7^6)q^3\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{14};q^{49}\right]_\infty}+(1+\zeta_7+\zeta_7^2+\zeta_7^5+\zeta_7^6)q^4\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{21};q^{49}\right]_\infty}\\&\quad-(\zeta_7^2+\zeta_7^5)q^{13}\frac{1}{\left(q^{49};q^{49}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{147n(n+1)/2}}{1-q^{49n+14}}+q^6\frac{\left(q^{49};q^{49}\right)_\infty\left[q^7;q^{49}\right]_\infty}{\left[q^{14},q^{21};q^{49}\right]_\infty},\end{aligned} \tag{2.8}$$

$$S_{F3}(\zeta_3,q)=q\frac{\left(q^{18};q^{18}\right)_\infty\left(q^9;q^9\right)_\infty}{\left(q^6;q^6\right)_\infty}+q^2\frac{\left(q^{18};q^{18}\right)_\infty^4\left(q^3;q^3\right)_\infty}{\left(q^9;q^9\right)_\infty^2\left(q^6;q^6\right)_\infty^2}, \tag{2.9}$$

$S_{F3}(\zeta_5,q)$

$$= q\frac{(q^{25};q^{25})_\infty}{[q^{10};q^{50}]_\infty} + \frac{3+\zeta_5+\zeta_5^4}{5}q^2\frac{(q^{50};q^{50})_\infty\,[q^{15};q^{50}]_\infty}{(q^{25};q^{50})_\infty\,[q^{10};q^{50}]_\infty} - \frac{1+2\zeta_5+2\zeta_5^4}{5}q^2\frac{(q^{25};q^{25})_\infty\,[q^{10};q^{25}]_\infty}{[q^{5};q^{25}]_\infty\,[q^{20};q^{50}]_\infty}$$
$$+ \frac{3+\zeta_5+\zeta_5^4}{5}q^2\frac{(q^{25};q^{25})_\infty\,[q^{5};q^{25}]_\infty}{[q^{10};q^{25}]_\infty\,[q^{10};q^{50}]_\infty} - \frac{1+2\zeta_5+2\zeta_5^4}{5}q^7\frac{(q^{50};q^{50})_\infty\,[q^{5};q^{50}]_\infty}{(q^{25};q^{50})_\infty\,[q^{20};q^{50}]_\infty}$$
$$+ (1+\zeta_5+\zeta_5^4)q^3\frac{(q^{25};q^{25})_\infty}{[q^{20};q^{50}]_\infty}, \tag{2.10}$$

$S_{F3}(\zeta_7,q)$

$$= \frac{18+9\zeta_7+3\zeta_7^2+3\zeta_7^5+9\zeta_7^6}{7}q\frac{(q^{98};q^{98})_\infty\,[q^{35};q^{98}]_\infty}{(q^{49};q^{98})_\infty\,[q^{14};q^{98}]_\infty} - \frac{5+6\zeta_7+2\zeta_7^2+2\zeta_7^5+6\zeta_7^6}{7}q\frac{(q^{14};q^{14})_\infty}{(q^{7};q^{14})_\infty}$$
$$+ \frac{2+\zeta_7-2\zeta_7^2-2\zeta_7^5+\zeta_7^6}{7}q^8\frac{(q^{98};q^{98})_\infty\,[q^{21};q^{98}]_\infty}{(q^{49};q^{98})_\infty\,[q^{28};q^{98}]_\infty} - \frac{2+\zeta_7-2\zeta_7^2-2\zeta_7^5+\zeta_7^6}{7}q\frac{(q^{49};q^{49})_\infty\,[q^{35};q^{98}]_\infty}{[q^{21};q^{49}]_\infty}$$
$$- \frac{4+2\zeta_7+3\zeta_7^2+3\zeta_7^5+2\zeta_7^6}{7}q\frac{(q^{49};q^{49})_\infty\,[q^{21};q^{98}]_\infty}{[q^{7};q^{49}]_\infty} + q^2\frac{(q^{49};q^{49})_\infty\,[q^{14};q^{49}]_\infty}{[q^{7};q^{49}]_\infty\,[q^{28};q^{98}]_\infty}$$
$$+ q^3(1+\zeta_7+\zeta_7^6)\frac{(q^{49};q^{49})_\infty\,[q^{7};q^{49}]_\infty}{[q^{21};q^{49}]_\infty\,[q^{14};q^{98}]_\infty} + q^5(1+\zeta_7+\zeta_7^2+\zeta_7^5+\zeta_7^6)\frac{(q^{49};q^{49})_\infty\,[q^{21};q^{49}]_\infty}{[q^{14};q^{49}]_\infty\,[q^{42};q^{98}]_\infty}, \tag{2.11}$$

$S_{G4}(\zeta_5,q)$

$$= -(1+\zeta_5+\zeta_5^4)q^{10}\frac{(q^{100};q^{100})_\infty\,[q^{10};q^{200}]_\infty}{[q^{10};q^{50}]_\infty\,[q^{5};q^{100}]_\infty} - (\zeta_5+\zeta_5^4)q^5\frac{(q^{50};q^{50})_\infty}{(q^{25};q^{50})_\infty\,[q^{20};q^{50}]_\infty}$$
$$- q^6\frac{(q^{100};q^{100})_\infty\,[q^{30};q^{200}]_\infty}{[q^{10};q^{50}]_\infty\,[q^{15};q^{100}]_\infty} - q^{12}\frac{(q^{100};q^{100})_\infty\,[q^{10};q^{200}]_\infty}{[q^{20};q^{50}]_\infty\,[q^{5};q^{100}]_\infty} - q^3\frac{(q^{50};q^{50})_\infty}{(q^{25};q^{50})_\infty\,[q^{10};q^{50}]_\infty}$$
$$- (\zeta_5+\zeta_5^4)q^8\frac{(q^{100};q^{100})_\infty\,[q^{30};q^{200}]_\infty}{[q^{20};q^{50}]_\infty\,[q^{15};q^{100}]_\infty}, \tag{2.12}$$

$S_{AG4}(\zeta_5,q)$

$$= -q^{10}\frac{(q^{100};q^{100})_\infty\,[q^{10};q^{200}]_\infty}{[q^{10};q^{50}]_\infty\,[q^{5};q^{100}]_\infty} - q\frac{(q^{100};q^{100})_\infty\,[q^{70};q^{200}]_\infty}{[q^{10};q^{50}]_\infty\,[q^{35};q^{100}]_\infty}$$
$$- (1+\zeta_5+\zeta_5^4)q^6\frac{(q^{100};q^{100})_\infty\,[q^{30};q^{200}]_\infty}{[q^{10};q^{50}]_\infty\,[q^{15};q^{100}]_\infty} - (\zeta_5+\zeta_5^4)q^{12}\frac{(q^{100};q^{100})_\infty\,[q^{10};q^{200}]_\infty}{[q^{20};q^{50}]_\infty\,[q^{5};q^{100}]_\infty}$$
$$- (\zeta_5+\zeta_5^4)q^3\frac{(q^{100};q^{100})_\infty\,[q^{70};q^{200}]_\infty}{[q^{20};q^{50}]_\infty\,[q^{35};q^{100}]_\infty} - q^8\frac{(q^{100};q^{100})_\infty\,[q^{30};q^{200}]_\infty}{[q^{20};q^{50}]_\infty\,[q^{15};q^{100}]_\infty}. \tag{2.13}$$

We note the identities of Theorems 2.2 and 2.3 are inherently different. In Theorem 2.3 we have an identity for $z=\zeta_\ell$, a primitive ℓ^{th} root of unity and we have an explicit formula for each term of the ℓ-dissection. In Theorem 2.2 we have an identity for general z, but if we set $z=\zeta_\ell$, we are able to determine some but not necessary all of the terms in the ℓ-dissection.

With these two Theorems we will show that the coefficients of the following terms are zero: q^{3n} in $S_{F3}(\zeta_3,q)$, q^{3n+2} in $S_{J1}(\zeta_3,q)$, q^{3n} in $S_{J2}(\zeta_3,q)$, q^{3n+1} in $S_{J3}(\zeta_3,q)$, q^{5n+1} in $S_{B2}(\zeta_5,q)$, q^{5n+4} in $S_{B2}(\zeta_5,q)$, q^{5n} in $S_{F3}(\zeta_5,q)$, q^{5n+4} in $S_{F3}(\zeta_5,q)$, q^{5n+4} in $S_{G4}(\zeta_5,q)$, q^{5n+4} in $S_{AG4}(\zeta_5,q)$, q^{7n+1} in $S_{B2}(\zeta_7,q)$, q^{7n+5} in $S_{B2}(\zeta_7,q)$, q^{7n} in $S_{F3}(\zeta_7,q)$, q^{7n+4} in $S_{F3}(\zeta_7,q)$, and q^{7n+6} in $S_{F3}(\zeta_7,q)$. As explained in the introduction, this gives the following corollary which also establishes the congruences of Theorem 2.1.

Corollary 2.4. *For $n\ge 0$,*

$$M_{F3}(0,3,3n) = M_{F3}(1,3,3n) = M_{F3}(2,3,3n) = \frac{1}{3}\mathrm{spt}_{F3}\left(3n\right),$$
$$M_{J1}(0,3,3n+2) = M_{J1}(1,3,3n+2) = M_{J1}(2,3,3n+2) = \frac{1}{3}\mathrm{spt}_{J1}\left(3n+2\right),$$
$$M_{J2}(0,3,3n) = M_{J2}(1,3,3n) = M_{J2}(2,3,3n) = \frac{1}{3}\mathrm{spt}_{J2}\left(3n\right),$$

$$
\begin{aligned}
M_{J3}(0,3,3n+1) &= M_{J3}(1,3,3n+1) = M_{J3}(2,3,3n+1) = \frac{1}{3}\mathrm{spt}_{J3}\left(3n+1\right),\\
M_{B2}(0,5,5n+1) &= M_{B2}(1,5,5n+1) = M_{B2}(2,5,5n+1) = M_{B2}(3,5,5n+1) = M_{B2}(4,5,5n+1)\\
&= \frac{1}{5}\mathrm{spt}_{B2}\left(5n+1\right),\\
M_{B2}(0,5,5n+4) &= M_{B2}(1,5,5n+4) = M_{B2}(2,5,5n+4) = M_{B2}(3,5,5n+4) = M_{B2}(4,5,5n+4)\\
&= \frac{1}{5}\mathrm{spt}_{B2}\left(5n+4\right),\\
M_{F3}(0,5,5n) &= M_{F3}(1,5,5n) = M_{F3}(2,5,5n) = M_{F3}(3,5,5n) = M_{F3}(4,5,5n)\\
&= \frac{1}{5}\mathrm{spt}_{F3}\left(5n\right),\\
M_{F3}(0,5,5n+4) &= M_{F3}(1,5,5n+4) = M_{F3}(2,5,5n+4) = M_{F3}(3,5,5n+4) = M_{F3}(4,5,5n+4)\\
&= \frac{1}{5}\mathrm{spt}_{F3}\left(5n+4\right),\\
M_{G4}(0,5,5n+4) &= M_{G4}(1,5,5n+4) = M_{G4}(2,5,5n+4) = M_{G4}(3,5,5n+4) = M_{G4}(4,5,5n+4)\\
&= \frac{1}{5}\mathrm{spt}_{G4}\left(5n+4\right),\\
M_{AG4}(0,5,5n+4) &= M_{AG4}(1,5,5n+4) = M_{AG4}(2,5,5n+4) = M_{AG4}(3,5,5n+4) = M_{AG4}(4,5,5n+4)\\
&= \frac{1}{5}\mathrm{spt}_{AG4}\left(5n+4\right),\\
M_{B2}(0,7,7n+1) &= M_{B2}(1,7,7n+1) = M_{B2}(2,7,7n+1) = M_{B2}(3,7,7n+1) = M_{B2}(4,7,7n+1)\\
&= M_{B2}(5,7,7n+1) = M_{B2}(6,7,7n+1) = \frac{1}{7}\mathrm{spt}_{B2}\left(7n+1\right),\\
M_{B2}(0,7,7n+5) &= M_{B2}(1,7,7n+5) = M_{B2}(2,7,7n+5) = M_{B2}(3,7,7n+5) = M_{B2}(4,7,7n+5)\\
&= M_{B2}(5,7,7n+5) = M_{B2}(6,7,7n+5) = \frac{1}{7}\mathrm{spt}_{B2}\left(7n+5\right),\\
M_{F3}(0,7,7n) &= M_{F3}(1,7,7n) = M_{F3}(2,7,7n) = M_{F3}(3,7,7n) = M_{F3}(4,7,7n)\\
&= M_{F3}(5,7,7n) = M_{F3}(6,7,7n) = \frac{1}{7}\mathrm{spt}_{F3}\left(7n\right),\\
M_{F3}(0,7,7n+4) &= M_{F3}(1,7,7n+4) = M_{F3}(2,7,7n+4) = M_{F3}(3,7,7n+4) = M_{F3}(4,7,7n+4)\\
&= M_{F3}(5,7,7n+4) = M_{F3}(6,7,7n+4) = \frac{1}{7}\mathrm{spt}_{F3}\left(7n+4\right),\\
M_{F3}(0,7,7n+6) &= M_{F3}(1,7,7n+6) = M_{F3}(2,7,7n+6) = M_{F3}(3,7,7n+6) = M_{F3}(4,7,7n+6)\\
&= M_{F3}(5,7,7n+6) = M_{F3}(6,7,7n+6) = \frac{1}{7}\mathrm{spt}_{F3}\left(7n+6\right).
\end{aligned}
$$

We note (2.1) follows from adding (2.2) and (2.3). Theorem 2.2 also lets us easily deduce the following product identities for $S_{F3}(z,q)$, $S_{G4}(z,q)$, and $S_{AG4}(z,q)$.

Corollary 2.5.

$$
\begin{aligned}
S_{F3}(z,q) &= \frac{\left(zq,z^{-1}q,q^2;q^2\right)_\infty}{\left(z,z^{-1},q;q^2\right)_\infty} - \frac{(q;q)_\infty}{\left(z,z^{-1};q^2\right)_\infty},\\
S_{G4}(z,q) &= \frac{z\left(-z^{-1}q,-zq^3,q^4;q^4\right)_\infty}{(1+z)\left(z,z^{-1};q^2\right)_\infty} + \frac{\left(-zq,-z^{-1}q^3,q^4;q^4\right)_\infty}{(1+z)\left(z,z^{-1};q^2\right)_\infty} - \frac{\left(q^2;q^2\right)_\infty}{\left(q,z,z^{-1};q^2\right)_\infty},\\
S_{AG4}(z,q) &= \frac{z\left(-zq,-z^{-1}q^3,q^4;q^4\right)_\infty}{(1+z)\left(z,z^{-1};q^2\right)_\infty} + \frac{\left(-z^{-1}q,-zq^3,q^4;q^4\right)_\infty}{(1+z)\left(z,z^{-1};q^2\right)_\infty} - \frac{\left(q^2;q^2\right)_\infty}{\left(q,z,z^{-1};q^2\right)_\infty}.
\end{aligned}
$$

These follow by rearranging the series in Theorem 2.2 and applying the Jacobi triple product identity. For example,

$$
\begin{aligned}
(1+z)\left(z,z^{-1},q;q^2\right)_\infty S_{F3}(z,q) &= \sum_{j=-\infty}^{\infty}(1-z^{j-1})(1-z^j)z^{1-j}(-1)^{j+1}q^{(j-1)^2}\\
&= \sum_{j=-\infty}^{\infty}(z^{1-j}+z^j)(-1)^{j+1}q^{(j-1)^2}-(1+z)\sum_{j=-\infty}^{\infty}(-1)^{j+1}q^{(j-1)^2}\\
&= \sum_{j=-\infty}^{\infty}(z^{j-1}+z^j)(-1)^{j+1}q^{(j-1)^2}-(1+z)\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\\
&= (1+z)\sum_{j=-\infty}^{\infty}z^j(-1)^jq^{j^2}-(1+z)\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\\
&= (1+z)\left(zq,z^{-1}q,q^2;q^2\right)_\infty-(1+z)\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.
\end{aligned}
$$

The identities for $S_{G4}(z,q)$ and $S_{AG4}(z,q)$ are similar.

We summarize the results of this article in the following table:

Bailey pair X	linear congruence mod p	single series identity for $S_X(z,q)$	product identity for $S_X(z,q)$	dissection identity for $S_X(\zeta_p,q)$
$B2$	$p=5,7$	No	No	Yes
$F3$	$p=3,5,7$	Yes	Yes	Yes
$G4$	$p=5$	Yes	Yes	Yes
$AG4$	$p=5$	Yes	Yes	Yes
$J1$	$p=3$	Yes	No	No
$J2$	$p=3$	Yes	No	No
$J3$	$p=3$	Yes	No	No

In Section 3 we prove the series identities in Theorem 2.2. In Section 4 we prove the dissections for $S_{B2}(\zeta_5,q)$ and $S_{B2}(\zeta_7,q)$. In Section 5 we prove the dissections for $S_{F3}(\zeta_3,q)$, $S_{F3}(\zeta_5,q)$, and $S_{F3}(\zeta_7,q)$. In Section 6 we sketch a proof that is independent of Theorem 2.2 for the dissections for $S_{AG4}(\zeta_5,q)$ and $S_{AG4}(\zeta_7,q)$. In Section 7 we use Theorems 2.2 and 2.3 to prove Corollary 2.4. In Section 8 we give some concluding remarks, in particular we discuss some additional Bailey pairs from [26] whose spt-crank-type functions reduce to previous functions after a change of variables.

3. Proof of Series Identities

The proof of these identities is to verify that the coefficients of each power of z on the left hand side and right hand side agree. This depends on a identity of Garvan from [17] to determine the coefficients of the powers of z in the left hand side of the identities in Theorem 2.2, and a variant of Bailey's lemma applied to one of two general Bailey pairs to transform the coefficients of the powers of z. The following is Proposition 4.1 of [17],

$$
\frac{(1+z)\left(z,z^{-1};q\right)_n}{(q;q)_{2n}} = \sum_{j=-n}^{n+1}\frac{(-1)^{j+1}(1-q^{2j-1})z^jq^{\frac{j(j-3)}{2}+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}. \tag{3.1}
$$

We recall a limiting case of Bailey's Lemma [8] gives that if (α,β) is a Bailey pair relative to (a,q) then

$$
\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n = \frac{(aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/(\rho_1\rho_2);q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\alpha_n}{(aq/\rho_1,aq/\rho_2;q)_n}.
$$

For one of the variants of Bailey's Lemma, we need the conjugate Bailey pair in the following lemma. We recall that a pair of sequences (δ, γ) is a conjugate Bailey pair relative to (a, q) if

$$\gamma_n = \sum_{j=n}^{\infty} \frac{\delta_j}{(q;q)_{j-n}(aq;q)_{j+n}}.$$

Different conjugate Bailey pairs give rise to different variants of Bailey's Lemma because Bailey's Transform states that if (α, β) is a Bailey pair relative to (a, q) and (δ, γ) is a conjugate Bailey pair relative to (a, q) then

$$\sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n.$$

Lemma 3.1. *The following is a conjugate Bailey pair relative to (a, q),*

$$\delta_n^1 = \left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_n q^n,$$

$$\gamma_n^1 = \frac{q^n \left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_\infty (1-(z+z^{-1})\sqrt{a}q^{n+\frac{1}{2}} + aq^{2n})}{(q, aq; q)_\infty (1-z\sqrt{a}q^{n-\frac{1}{2}})(1-z\sqrt{a}q^{n+\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n-\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n+\frac{1}{2}})}.$$

With $z = \omega$, a primitive third root of unity, and $a \neq q$ this conjugate Bailey pair becomes

$$\delta_n^2 = \frac{\left(a^{\frac{3}{2}}q^{-\frac{3}{2}}; q^3\right)_n q^n}{\left(\sqrt{a}q^{-\frac{1}{2}}; q\right)_n}, \qquad \gamma_n^2 = \frac{q^n \left(a^{\frac{3}{2}}q^{-\frac{3}{2}}; q^3\right)_\infty (1-\sqrt{a}q^{n-\frac{1}{2}})(1-\sqrt{a}q^{n+\frac{1}{2}})\left(1+\sqrt{a}q^{n+\frac{1}{2}} + aq^{2n}\right)}{\left(q, aq, \sqrt{a}q^{-\frac{1}{2}}; q\right)_\infty (1-a^{\frac{3}{2}}q^{3n-\frac{3}{2}})(1-a^{\frac{3}{2}}q^{3n+\frac{3}{2}})}.$$

Proof. We are to show that

$$\frac{q^n \left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_\infty (1-(z+z^{-1})\sqrt{a}q^{n+\frac{1}{2}} + aq^{2n})}{(q, aq; q)_\infty (1-z\sqrt{a}q^{n-\frac{1}{2}})(1-z\sqrt{a}q^{n+\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n-\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n+\frac{1}{2}})}$$
$$= \sum_{j=n}^{\infty} \frac{\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_j q^j}{(q;q)_{j-n}(aq;q)_{j+n}}.$$

Other than elementary rearrangements, we only need Heine's Transformation, which can be found as Corollary 2.3 in [1]. We recall Heine's Transformation is

$${}_2\phi_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; q, \ z\right) = \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1\left(\begin{matrix} abz/c, \ b \\ bz \end{matrix}; q, \ c/b\right).$$

We have

$$\sum_{j=n}^{\infty} \frac{\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_j q^j}{(q;q)_{j-n}(aq;q)_{j+n}}$$
$$= \sum_{j=0}^{\infty} \frac{\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_{j+n} q^{j+n}}{(q;q)_j (aq;q)_{j+2n}}$$
$$= \frac{q^n \left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_n}{(aq;q)_{2n}} \sum_{j=0}^{\infty} \frac{\left(z\sqrt{a}q^{n-\frac{1}{2}}, z^{-1}\sqrt{a}q^{n-\frac{1}{2}}; q\right)_j q^j}{(q;q)_j (aq^{2n+1};q)_j}$$
$$= \frac{q^n \left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_n}{(aq;q)_{2n}} {}_2\phi_1\left(\begin{matrix} z\sqrt{a}q^{n-\frac{1}{2}}, \ z^{-1}\sqrt{a}q^{n-\frac{1}{2}} \\ aq^{2n+1} \end{matrix}; q, \ q\right)$$
$$= \frac{q^n \left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}}; q\right)_n}{(aq;q)_{2n}} \frac{\left(z\sqrt{a}q^{n+\frac{3}{2}}, z^{-1}\sqrt{a}q^{n+\frac{1}{2}}; q\right)_\infty}{(aq^{2n+1}, q; q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{-1}, \ z^{-1}\sqrt{a}q^{n-\frac{1}{2}} \\ z^{-1}\sqrt{a}q^{n+\frac{1}{2}} \end{matrix}; q, \ z\sqrt{a}q^{n+\frac{3}{2}}\right)$$

$$= \frac{q^n\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}};q\right)_\infty}{(aq,q;q)_\infty(1-z\sqrt{a}q^{n-\frac{1}{2}})(1-z\sqrt{a}q^{n+\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n-\frac{1}{2}})}\left(1+\frac{(1-q^{-1})(1-z^{-1}\sqrt{a}q^{n-\frac{1}{2}})z\sqrt{a}q^{n+\frac{3}{2}}}{(1-q)(1-z^{-1}\sqrt{a}q^{n+\frac{1}{2}})}\right)$$
$$= \frac{q^n\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}};q\right)_\infty\left(1-(z+z^{-1})\sqrt{a}q^{n+\frac{1}{2}}+aq^{2n}\right)}{(aq,q;q)_\infty(1-z\sqrt{a}q^{n-\frac{1}{2}})(1-z\sqrt{a}q^{n+\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n-\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n+\frac{1}{2}})}.$$

□

Lemma 3.2. *If (α,β) is a Bailey pair relative to (a,q) then*

$$\sum_{n=0}^{\infty}\left(\sqrt{a};q\right)_n(-1)^n a^{\frac{n}{2}}q^{\frac{n(n+1)}{2}}\beta_n = \frac{(\sqrt{a}q;q)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-\sqrt{a})(-1)^n a^{\frac{n}{2}}q^{\frac{n(n+1)}{2}}\alpha_n}{(1-\sqrt{a}q^n)}, \tag{3.2}$$

$$\sum_{n=0}^{\infty}\left(\sqrt{aq};q\right)_n(-1)^n a^{\frac{n}{2}}q^{\frac{n^2}{2}}\beta_n = \frac{\left(\sqrt{aq};q\right)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{\frac{n}{2}}q^{\frac{n^2}{2}}\alpha_n, \tag{3.3}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}};q\right)_n q^n\beta_n\\ &= \frac{\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}};q\right)_\infty}{(q,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-(z+z^{-1})\sqrt{a}q^{n+\frac{1}{2}}+aq^{2n})q^n\alpha_n}{(1-z\sqrt{a}q^{n-\frac{1}{2}})(1-z\sqrt{a}q^{n+\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n-\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n+\frac{1}{2}})},\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}};q\right)_n q^{2n}\beta_n\\ &= \frac{\left(z\sqrt{a}q^{-\frac{1}{2}}, z^{-1}\sqrt{a}q^{-\frac{1}{2}};q\right)_\infty}{(q,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-q)q^{2n}\alpha_n}{(1-z\sqrt{a}q^{n-\frac{1}{2}})(1-z\sqrt{a}q^{n+\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n-\frac{1}{2}})(1-z^{-1}\sqrt{a}q^{n+\frac{1}{2}})},\end{aligned} \tag{3.5}$$

$$\sum_{n=0}^{\infty}\frac{\left(a^{\frac{3}{2}}q^{-\frac{3}{2}};q^3\right)_n q^n\beta_n}{\left(\sqrt{a}q^{-\frac{1}{2}};q\right)_n} = \frac{\left(a^{\frac{3}{2}}q^{-\frac{3}{2}};q^3\right)_\infty}{\left(q,aq,\sqrt{a}q^{-\frac{1}{2}};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(1-\sqrt{a}q^{n-\frac{1}{2}})(1-\sqrt{a}q^{n+\frac{1}{2}})(1+\sqrt{a}q^{n+\frac{1}{2}}+aq^{2n})q^n\alpha_n}{(1-a^{\frac{3}{2}}q^{3n-\frac{3}{2}})(1-a^{\frac{3}{2}}q^{3n+\frac{3}{2}})}, \tag{3.6}$$

$$\sum_{n=0}^{\infty}\frac{\left(a^{\frac{3}{2}}q^{-\frac{3}{2}};q^3\right)_n q^{2n}\beta_n}{\left(\sqrt{a}q^{-\frac{1}{2}};q\right)_n} = \frac{\left(a^{\frac{3}{2}}q^{-\frac{3}{2}};q^3\right)_\infty}{\left(q,aq,\sqrt{a}q^{-\frac{1}{2}};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(1-q)(1-\sqrt{a}q^{n-\frac{1}{2}})(1-\sqrt{a}q^{n+\frac{1}{2}})q^{2n}\alpha_n}{(1-a^{\frac{3}{2}}q^{3n-\frac{3}{2}})(1-a^{\frac{3}{2}}q^{3n+\frac{3}{2}})}. \tag{3.7}$$

If (α,β) is a Bailey pair relative to (a,q^2) then

$$\sum_{n=0}^{\infty}\left(-\sqrt{a};q\right)_{2n}q^n\beta_n = \frac{(-\sqrt{a}q;q)_\infty}{(aq^2,q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1+\sqrt{a})q^n\alpha_n}{(1+\sqrt{a}q^{2n})}. \tag{3.8}$$

Proof. Equation (3.2) follows from Bailey's Lemma by letting $\rho_1=\sqrt{a}$ and $\rho_2\to\infty$. Equation (3.3) follows from Bailey's Lemma by letting $\rho_1=\sqrt{aq}$ and $\rho_2\to\infty$. Equation (3.8) follows from Bailey's Lemma by letting $q\mapsto q^2$, $\rho_1=-\sqrt{a}$, and $\rho_2=-q\sqrt{a}$. Equations (3.4) and (3.6) are Bailey's Transform with the conjugate Bailey pairs of Lemma 3.1. Equation (3.5) follows from Bailey's Lemma by letting $\rho_1=z\sqrt{a}q^{-\frac{1}{2}}$ and $\rho_2=z^{-1}\sqrt{a}q^{-\frac{1}{2}}$. Lastly, equation (3.7) follows by letting $z=\omega$, a primitive third root of unity, in (3.5). □

We use the following Bailey pairs relative to (a,q),

$$\beta_n^*(a,q)=\frac{1}{(aq,q;q)_n}, \qquad \alpha_n^*(a,q)=\begin{cases}1 & n=0\\ 0 & n\geq 1\end{cases}, \tag{3.9}$$

$$\beta_n^{**}(a,q) = \frac{1}{(aq^2,q;q)_n}, \qquad \alpha_n^{**}(a,q) = \begin{cases} 1 & n=0 \\ -aq & n=1 \\ 0 & n\geq 2 \end{cases}. \tag{3.10}$$

That these are Bailey pairs relative to (a,q) follows immediately from the definition of a Bailey pair.

Proof of (2.2). By (3.1) we have that

$$\begin{aligned}(1+z)\left(z,z^{-1};q\right)_\infty S_{J2}(z,q) &= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{\left(q^3;q^3\right)_{n-1} q^n(1+z)\left(z,z^{-1};q\right)_n}{(q;q)_{2n}(q;q)_{n-1}} \\ &= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{\left(q^3;q^3\right)_{n-1} q^n}{(q;q)_{n-1}}\sum_{j=-n}^{n+1}\frac{(-1)^{j+1}(1-q^{2j-1})z^j q^{\frac{j(j-3)}{2}+1}}{(q;q)_{n+j}(q;q)_{n-j+1}}.\end{aligned}$$

We note the coefficients of z^{-j} and z^{j+1} are then the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{J2}(z,q)$, so we need only determine the coefficients of z^j for $j\geq 1$. For $j\geq 2$ we see the coefficient of z^j in $(1+z)\left(z,z^{-1};q\right)_\infty S_{J2}(z,q)$ is given by

$$\begin{aligned}&\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=j-1}^{\infty}\frac{\left(q^3;q^3\right)_{n-1}(-1)^{j+1}(1-q^{2j-1})q^{n+\frac{j(j-3)}{2}+1}}{(q;q)_{n-1}(q;q)_{n+j}(q;q)_{n-j+1}} \\ &= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{\left(q^3;q^3\right)_{n+j-2}(-1)^{j+1}(1-q^{2j-1})q^{n+\frac{j(j-1)}{2}}}{(q;q)_{n+j-2}(q;q)_{n+2j-1}(q;q)_n} \\ &= \frac{(q;q)_\infty^2(-1)^{j+1}\left(q^3;q^3\right)_{j-2}(1-q^{2j-1})q^{\frac{j(j-1)}{2}}}{(q^3;q^3)_\infty(q;q)_{j-2}(q;q)_{2j-1}}\sum_{n=0}^{\infty}\frac{\left(q^{3j-3};q^3\right)_n q^n}{(q^{j-1};q)_n(q^{2j};q)_n(q;q)_n} \\ &= \frac{(q;q)_\infty^2(-1)^{j+1}\left(q^3;q^3\right)_{j-2}(1-q^{2j-1})q^{\frac{j(j-1)}{2}}}{(q^3;q^3)_\infty(q;q)_{j-2}(q;q)_{2j-1}}\sum_{n=0}^{\infty}\frac{\left(q^{3j-3};q^3\right)_n q^n\beta_n^*(q^{2j-1},q)}{(q^{j-1};q)_n}.\end{aligned}$$

We now apply (3.6) so that the coefficient of z^j in $(1+z)\left(z,z^{-1};q\right)_\infty S_{J2}(z,q)$ is given by

$$\begin{aligned}&\frac{(q;q)_\infty^2(-1)^{j+1}\left(q^3;q^3\right)_{j-2}(1-q^{2j-1})q^{\frac{j(j-1)}{2}}\left(q^{3j-3};q^3\right)_\infty(1-q^{j-1})(1-q^j)(1+q^j+q^{2j-1})}{(q^3;q^3)_\infty(q;q)_{j-2}(q;q)_{2j-1}(q,q^{2j},q^{j-1};q)_\infty(1-q^{3j-3})(1-q^{3j})} \\ &= \frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{j(j-1)}{2}}(1-q^{j-1})(1-q^j)(1+q^j+q^{2j-1})}{(q;q)_\infty(1-q^{3j-3})(1-q^{3j})} \\ &= \frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{j(j-1)}{2}}(1+q^j+q^{2j-1})}{(q;q)_\infty(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)}.\end{aligned}$$

The calculations are similar for the coefficient of z, except that we use (3.4). In particular, we have that the coefficient of z in $(1+z)\left(z,z^{-1};q\right)_\infty S_{J2}(z,q)$ is given by

$$\begin{aligned}&\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{\left(q^3;q^3\right)_{n-1}q^n(1-q)}{(q;q)_{n-1}(q;q)_{n+1}(q;q)_n} \\ &= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty}\frac{\left(\omega q,\omega^{-1}q;q\right)_{n-1}q^n(1-q)}{(q;q)_{n+1}(q;q)_n} \\ &= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{\left(\omega q,\omega^{-1}q;q\right)_{n-1}q^n(1-q)}{(q;q)_{n+1}(q;q)_n} - \frac{(q;q)_\infty^2}{3\,(q^3;q^3)_\infty} \\ &= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty(1-\omega)(1-\omega^{-1})}\sum_{n=0}^{\infty}\frac{\left(\omega,\omega^{-1};q\right)_n q^n}{(q^2;q)_n(q;q)_n} - \frac{(q;q)_\infty^2}{3\,(q^3;q^3)_\infty}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty (1-\omega)(1-\omega^{-1})} \sum_{n=0}^{\infty} \left(\omega,\omega^{-1};q\right)_n q^n \beta_n^*(q,q) - \frac{(q;q)_\infty^2}{3\,(q^3;q^3)_\infty} \\
&= \frac{(q;q)_\infty^2 \left(\omega,\omega^{-1};q\right)_\infty (1+2q)}{(q^3;q^3)_\infty (q^2,q;q)_\infty (1-\omega)^2(1-\omega^{-1})^2(1-\omega q)(1-\omega^{-1}q)} - \frac{(q;q)_\infty^2}{3\,(q^3;q^3)_\infty} \\
&= \frac{(1-q)(1+2q)}{(q;q)_\infty (1-\omega)(1-\omega^{-1})(1-\omega q)(1-\omega^{-1}q)} - \frac{(q;q)_\infty^2}{3\,(q^3;q^3)_\infty}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty S_{J2}(z,q) \\
&= -\frac{(1+z)}{3}\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty} + \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty} \frac{(z^j+z^{1-j})(-1)^{j+1} q^{\frac{j(j-1)}{2}}(1-q^{2j-1})(1+q^j+q^{2j-1})}{(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)}.
\end{aligned}
$$

Here setting $z=1$ yields

$$
\frac{1}{3}\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty} = \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j+1} q^{\frac{j(j-1)}{2}}(1-q^{2j-1})(1+q^j+q^{2j-1})}{(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)},
$$

so in fact

$$
\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty S_{J2}(z,q) \\
&= \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty} \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1} q^{\frac{j(j-1)}{2}}(1-q^{2j-1})(1+q^j+q^{2j-1})}{(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)} \\
&= \frac{1}{(q;q)_\infty}\sum_{j=2}^{\infty} \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1} q^{\frac{j(j-1)}{2}}(1-q^{2j-1})(1+q^j+q^{2j-1})}{(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)} \\
&= \frac{1}{(q;q)_\infty}\sum_{j=2}^{\infty} \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1} q^{\frac{j(j-1)}{2}}(1-q^{2j-1})(1+q^j+q^{2j-1})(1-q^{j-1})(1-q^j)}{(1-q^{3j-3})(1-q^{3j})} \\
&= \frac{1}{(q;q)_\infty}\sum_{j=2}^{\infty} \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1} q^{\frac{j(j-1)}{2}}(1-q^{j-1}-q^{2j}+q^{4j-1}+q^{5j-3}-q^{6j-3})}{(1-q^{3j-3})(1-q^{3j})}.
\end{aligned}
$$

□

Proof of (2.3). By (3.1) we have that

$$
\begin{aligned}
(1+z)\left(z,z^{-1};q\right)_\infty S_{J3}(z,q) &= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty} \frac{\left(q^3;q^3\right)_{n-1} q^{2n}(1+z)\left(z,z^{-1};q\right)_n}{(q;q)_{2n}\,(q;q)_{n-1}} \\
&= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=1}^{\infty} \frac{\left(q^3;q^3\right)_{n-1} q^{2n}}{(q;q)_{n-1}} \sum_{j=-n}^{n+1} \frac{(-1)^{j+1}(1-q^{2j-1})z^j q^{\frac{j(j-3)}{2}+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}.
\end{aligned}
$$

We note the coefficients of z^{-j} and z^{j+1} are then the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{J3}(z,q)$, so we need only determine the coefficients of z^j for $j \ge 1$. The proof is now the same as it was for $S_{J2}(z,q)$, except that we use (3.7). For $j \ge 2$ we see the coefficient of z^j in $(1+z)\left(z,z^{-1};q\right)_\infty S_{J3}(z,q)$ is given by

$$
\begin{aligned}
&\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=j-1}^{\infty} \frac{\left(q^3;q^2\right)_{n-1}(-1)^{j+1}(1-q^{2j-1})q^{2n+\frac{j(j-3)}{2}+1}}{(q;q)_{n-1}\,(q;q)_{n+j}\,(q;q)_{n-j+1}} \\
&= \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}\sum_{n=0}^{\infty} \frac{\left(q^3;q^3\right)_{n+j-2}(-1)^{j+1}(1-q^{2j-1})q^{2n+\frac{j(j+1)}{2}-1}}{(q;q)_{n+j-2}\,(q;q)_{n+2j-1}\,(q;q)_n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(q;q)_\infty^2 (-1)^{j+1} \left(q^3;q^3\right)_{j-2} (1-q^{2j-1}) q^{\frac{j(j+1)}{2}-1}}{\left(q^3;q^3\right)_\infty (q;q)_{j-2} (q;q)_{2j-1}} \sum_{n=0}^\infty \frac{\left(q^{3j-3};q^3\right)_n q^{2n}}{\left(q^{j-1};q\right)_n \left(q^{2j};q\right)_n (q;q)_n} \\
&= \frac{(q;q)_\infty^2 (-1)^{j+1} \left(q^3;q^3\right)_{j-2} (1-q^{2j-1}) q^{\frac{j(j+1)}{2}-1}}{\left(q^3;q^3\right)_\infty (q;q)_{j-2} (q;q)_{2j-1}} \sum_{n=0}^\infty \frac{\left(q^{3j-3};q^3\right)_n q^{2n} \beta_n^*(q^{2j-1},q)}{\left(q^{j-1};q\right)_n} \\
&= \frac{(q;q)_\infty^2 (-1)^{j+1} \left(q^3;q^3\right)_{j-2} (1-q^{2j-1}) q^{\frac{j(j+1)}{2}-1} \left(q^{3j-3};q^3\right)_\infty (1-q)(1-q^{j-1})(1-q^j)}{\left(q^3;q^3\right)_\infty (q;q)_{j-2} (q;q)_{2j-1} \left(q,q^{2j},q^{j-1};q\right)_\infty (1-q^{3j-3})(1-q^{3j})} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})(1-q)(1-q^{j-1})(1-q^j) q^{\frac{j(j+1)}{2}-1}}{(q;q)_\infty (1-q^{3j-3})(1-q^{3j})} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})(1-q) q^{\frac{j(j+1)}{2}-1}}{(q;q)_\infty (1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega q^{j-1})}.
\end{aligned}
$$

The calculations are similar for the coefficient of z, but we use (3.5). In particular, we have that the coefficient of z in $(1+z)\left(z,z^{-1};q\right)_\infty S_{J3}(z,q)$ is given by

$$
\begin{aligned}
&\frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty} \sum_{n=1}^\infty \frac{\left(q^3;q^3\right)_{n-1} q^{2n}(1-q)}{(q;q)_{n-1} (q;q)_{n+1} (q;q)_n} \\
&= \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty} \sum_{n=1}^\infty \frac{\left(\omega q,\omega^{-1}q;q\right)_{n-1} q^{2n}(1-q)}{(q;q)_{n+1} (q;q)_n} \\
&= \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty} \sum_{n=0}^\infty \frac{\left(\omega q,\omega^{-1}q;q\right)_{n-1} q^{2n}(1-q)}{(q;q)_{n+1} (q;q)_n} - \frac{(q;q)_\infty^2}{3\left(q^3;q^3\right)_\infty} \\
&= \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty (1-\omega)(1-\omega^{-1})} \sum_{n=0}^\infty \frac{\left(\omega,\omega^{-1};q\right)_n q^{2n}}{\left(q^2;q\right)_n (q;q)_n} - \frac{(q;q)_\infty^2}{3\left(q^3;q^3\right)_\infty} \\
&= \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty (1-\omega)(1-\omega^{-1})} \sum_{n=0}^\infty \left(\omega,\omega^{-1};q\right)_n q^{2n} \beta_n^*(q,q) - \frac{(q;q)_\infty^2}{3\left(q^3;q^3\right)_\infty} \\
&= \frac{(q;q)_\infty^2 \left(\omega,\omega^{-1};q\right)_\infty (1-q)}{\left(q^3;q^3\right)_\infty \left(q,q^2;q\right)_\infty (1-\omega)^2(1-\omega^{-1})^2(1-\omega q)(1-\omega^{-1}q)} - \frac{(q;q)_\infty^2}{3\left(q^3;q^3\right)_\infty} \\
&= \frac{(1-q)^2}{(q;q)_\infty (1-\omega)(1-\omega^{-1})(1-\omega q)(1-\omega^{-1}q)} - \frac{(q;q)_\infty^2}{3\left(q^3;q^3\right)_\infty}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty S_{J3}(z,q) \\
&= -\frac{(1+z)}{3} \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty} + \frac{1}{(q;q)_\infty} \sum_{j=1}^\infty \frac{(z^j+z^{1-j})(-1)^{j+1} q^{\frac{j(j+1)}{2}-1}(1-q^{2j-1})(1-q)}{(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)}.
\end{aligned}
$$

Here setting $z=1$ yields

$$
\frac{1}{3} \frac{(q;q)_\infty^2}{\left(q^3;q^3\right)_\infty} = \frac{1}{(q;q)_\infty} \sum_{j=1}^\infty \frac{(-1)^{j+1} q^{\frac{j(j+1)}{2}-1}(1-q^{2j-1})(1-q)}{(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)},
$$

so in fact

$$
\begin{aligned}
&(1+z)\left(z,z^{-1},q;q\right)_\infty S_{J3}(z,q) \\
&= \sum_{j=1}^\infty \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1} q^{\frac{j(j+1)-1}{2}}(1-q^{2j-1})(1-q)}{(1-\omega q^{j-1})(1-\omega^{-1}q^{j-1})(1-\omega q^j)(1-\omega^{-1}q^j)}
\end{aligned}
$$

$$= \sum_{j=2}^{\infty} \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j+1)}{2}-1}(1-q-q^{j-1}-q^{j+1}+q^{3j-2}-q^{4j-2}-q^{3j}+q^{4j-1})}{(1-q^{3j-3})(1-q^{3j})}$$
$$= \sum_{j=2}^{\infty} \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}(q^{j-1}-q^j-q^{2j-2}+q^{2j}+q^{4j-3}-q^{4j-1}-q^{5j-3}+q^{5j-2})}{(1-q^{3j-3})(1-q^{3j})}.$$

□

Proof of (2.4). By (3.1) we have that

$$\begin{aligned}
(1+z)\left(z,z^{-1};q^2\right)_\infty S_{F3}(z,q) &= (q;q)_\infty \sum_{n=1}^{\infty} \frac{(1+z)\left(z,z^{-1};q^2\right)_n q^n}{(q;q)_{2n}} \\
&= (q;q)_\infty \sum_{n=1}^{\infty} \frac{(1+z)\left(z,z^{-1};q^2\right)_n q^n (-q;q)_{2n}}{(q^2;q^2)_{2n}} \\
&= (q;q)_\infty \sum_{n=1}^{\infty} q^n (-q;q)_{2n} \sum_{j=-n}^{n+1} \frac{(-1)^{j+1}(1-q^{4j-2})z^j q^{j(j-3)+2}}{(q^2;q^2)_{n+j}(q^2;q^2)_{n-j+1}}.
\end{aligned}$$

We note the coefficients of z^{-j} and z^{j+1} are then the same in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{F3}(z,q)$, so we need only determine the coefficients of z^j for $j \geq 1$. This time we will use (3.8). For $j \geq 2$, the coefficient of z^j in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{F3}(z,q)$ is given by

$$\begin{aligned}
&(q;q)_\infty \sum_{n=j-1}^{\infty} \frac{q^n (-q;q)_{2n} (-1)^{j+1}(1-q^{4j-2})q^{j(j-3)+2}}{(q^2;q^2)_{n+j}(q^2;q^2)_{n-j+1}} \\
&= (q;q)_\infty (-1)^{j+1}(1-q^{4j-2})q^{j(j-3)+2} \sum_{n=0}^{\infty} \frac{q^{n+j-1}(-q;q)_{2n+2j-2}}{(q^2;q^2)_{n+2j-1}(q^2;q^2)_n} \\
&= \frac{(q;q)_\infty (-1)^{j+1}(1-q^{4j-2})q^{(j-1)^2}(-q;q)_{2j-2}}{(q^2;q^2)_{2j-1}} \sum_{n=0}^{\infty} \frac{q^n \left(-q^{2j-1};q\right)_{2n}}{(q^{4j};q^2)_n (q^2;q^2)_n} \\
&= \frac{(q;q)_\infty (-1)^{j+1}(1-q^{4j-2})q^{(j-1)^2}(-q;q)_{2j-2}}{(q^2;q^2)_{2j-1}} \sum_{n=0}^{\infty} q^n \left(-q^{2j-1};q\right)_{2n} \beta_n^*(q^{4j-2},q^2) \\
&= \frac{(q;q)_\infty (-1)^{j+1}(1-q^{4j-2})q^{(j-1)^2}(-q;q)_{2j-2}\left(-q^{2j};q\right)_\infty}{(q^2;q^2)_{2j-1}(q^{4j},q;q^2)_\infty} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{(j-1)^2}}{(q;q^2)_\infty}.
\end{aligned}$$

The calculations for the coefficient of z are similar and we still use (3.8). In particular, the coefficient of z in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{F3}(z,q)$ is given by

$$\begin{aligned}
(q;q)_\infty \sum_{n=1}^{\infty} \frac{q^n (-q;q)_{2n}(1-q^2)}{(q^2;q^2)_{n+1}(q^2;q^2)_n} &= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n(-q;q)_{2n}}{(q^4;q^2)_n(q^2;q^2)_n} - (q;q)_\infty \\
&= (q;q)_\infty \sum_{n=0}^{\infty} q^n(-q;q)_{2n}\beta_n^*(q^2,q^2) - (q;q)_\infty \\
&= \frac{(q;q)_\infty \left(-q^2;q\right)_\infty}{(q^4,q;q^2)_\infty} \\
&= \frac{(1-q)}{(q;q^2)_\infty}.
\end{aligned}$$

Thus

$$(1+z)\left(z,z^{-1},q;q^2\right)_\infty S_{F3}(z,q)$$

$$= -(1+z)\left(q;q\right)_\infty \left(q;q^2\right)_\infty + \sum_{j=1}^{\infty}(z^j+z^{1-j})(-1)^{j+1}(1-q^{2j-1})q^{(j-1)^2}$$
$$= -(1+z)\left(q;q\right)_\infty \left(q;q^2\right)_\infty + \sum_{j=1}^{\infty}(z^j+z^{1-j})(-1)^{j+1}q^{(j-1)^2} + \sum_{j=1}^{\infty}(z^j+z^{1-j})(-1)^{j+1}q^{j^2}$$
$$= -(1+z)\left(q;q\right)_\infty \left(q;q^2\right)_\infty + \sum_{j=-\infty}^{\infty}(z^j+z^{1-j})(-1)^{j+1}q^{(j-1)^2}.$$

However, we have by Gauss that

$$\left(q;q\right)_\infty \left(q;q^2\right)_\infty = \sum_{j=-\infty}^{\infty}(-1)^j q^{j^2} = \sum_{j=-\infty}^{\infty}(-1)^{j+1}q^{(j-1)^2}.$$

We then have that

$$(1+z)\left(z,z^{-1},q;q^2\right)_\infty S_{F3}(z,q) = \sum_{j=-\infty}^{\infty}(z^j+z^{1-j}-1-z)(-1)^{j+1}q^{(j-1)^2}$$
$$= \sum_{j=-\infty}^{\infty}(1-z^{j-1})(1-z^j)z^{1-j}(-1)^{j+1}q^{(j-1)^2}.$$

□

Proof of (2.5). By (3.1) we have that

$$(1+z)\left(z,z^{-1};q^2\right)_\infty S_{G4}(z,q) = \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{(1+z)\left(z,z^{-1};q^2\right)_n(-1)^n q^{n^2+2n}}{\left(-q;q^2\right)_n\left(q^4;q^4\right)_n}$$
$$= \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{(1+z)\left(z,z^{-1};q^2\right)_n(-1)^n q^{n^2+2n}\left(q;q^2\right)_n}{\left(q^2;q^2\right)_{2n}}$$
$$= \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=1}^{\infty}(-1)^n q^{n^2+2n}\left(q;q^2\right)_n\sum_{j=-n}^{n+1}\frac{(-1)^{j+1}(1-q^{4j-2})z^j q^{j(j-3)+2}}{\left(q^2;q^2\right)_{n+j}\left(q^2;q^2\right)_{n-j+1}}.$$

We note the coefficients of z^{-j} and z^{j+1} are then the same in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{G4}(z,q)$, so we need only determine the coefficients of z^j for $j\geq 1$. This time we will use (3.2). For $j\geq 2$, the coefficient of z^j in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{G4}(z,q)$ is given by

$$\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=j-1}^{\infty}\frac{(-1)^{j+n+1}q^{n^2+2n}\left(q;q^2\right)_n(1-q^{4j-2})q^{j(j-3)+2}}{\left(q^2;q^2\right)_{n+j}\left(q^2;q^2\right)_{n-j+1}}$$
$$= \frac{\left(q^2;q^2\right)_\infty(1-q^{4j-2})q^{2j^2-3j+1}}{\left(q;q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2jn}\left(q;q^2\right)_{n+j-1}}{\left(q^2;q^2\right)_{n+2j-1}\left(q^2;q^2\right)_n}$$
$$= \frac{\left(q^2;q^2\right)_\infty\left(q;q^2\right)_{j-1}(1-q^{4j-2})q^{2j^2-3j+1}}{\left(q;q^2\right)_\infty\left(q^2;q^2\right)_{2j-1}}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2jn}\left(q^{2j-1};q^2\right)_n}{\left(q^{4j};q^2\right)_n\left(q^2;q^2\right)_n}$$
$$= \frac{\left(q^2;q^2\right)_\infty\left(q;q^2\right)_{j-1}(1-q^{4j-2})q^{2j^2-3j+1}}{\left(q;q^2\right)_\infty\left(q^2;q^2\right)_{2j-1}}\sum_{n=0}^{\infty}(-1)^n q^{n^2+2jn}\left(q^{2j-1};q^2\right)_n\beta_n^*(q^{4j-2},q^2)$$
$$= \frac{\left(q^2;q^2\right)_\infty\left(q;q^2\right)_{j-1}(1-q^{4j-2})q^{2j^2-3j+1}}{\left(q;q^2\right)_\infty\left(q^2;q^2\right)_{2j-1}}\frac{\left(q^{2j+1};q^2\right)_\infty}{\left(q^{4j};q^2\right)_\infty}$$
$$= (1+q^{2j-1})q^{2j^2-3j+1}.$$

The calculations for the coefficient of z are similar and we still use (3.2). In particular, the coefficient of z in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{G4}(z,q)$ is given by

$$\begin{aligned}\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2+2n}\left(q;q^2\right)_n(1-q^2)}{(q^2;q^2)_{n+1}\left(q^2;q^2\right)_n} &= \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2n}\left(q;q^2\right)_n}{\left(q^4;q^2\right)_n\left(q^2;q^2\right)_n} - \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\\ &= \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n^2+2n}\left(q;q^2\right)_n\beta_n^*(q^2;q^2) - \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\\ &= \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\frac{\left(q^3;q^2\right)_\infty}{\left(q^4;q^2\right)_\infty} - \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\\ &= 1+q-\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}.\end{aligned}$$

Thus

$$\begin{aligned}(1+z)\left(z,z^{-1};q^2\right)_\infty S_{G4}(z,q) &= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}+\sum_{j=1}^{\infty}(z^j+z^{1-j})(1+q^{2j-1})q^{2j^2-3j+1}\\ &= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}+\sum_{j=-\infty}^{\infty}z^j(1+q^{2j-1})q^{2j^2-3j+1}\\ &= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}+\sum_{j=-\infty}^{\infty}z^jq^{2j^2-3j+1}+\sum_{j=-\infty}^{\infty}z^jq^{2j^2-j}\\ &= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}+\sum_{j=-\infty}^{\infty}z^{1-j}q^{2j^2-j}+\sum_{j=-\infty}^{\infty}z^jq^{2j^2-j}\\ &= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}+\sum_{j=-\infty}^{\infty}(z^j+z^{1-j})q^{2j^2-j}.\end{aligned}$$

However, we have by Gauss that

$$\begin{aligned}\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty} &= \frac{1}{2}\sum_{n=-\infty}^{\infty}q^{n(n+1)/2}\\ &= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty}q^{n(2n+1)}+\sum_{n=-\infty}^{\infty}q^{(2n-1)n}\right)\\ &= \sum_{n=-\infty}^{\infty}q^{2n^2-n}.\end{aligned}\tag{3.11}$$

We then have that

$$\begin{aligned}(1+z)\left(z,z^{-1};q^2\right)_\infty S_{G4}(z,q) &= \sum_{j=-\infty}^{\infty}(z^j+z^{1-j}-1-z)q^{2j^2-j}\\ &= \sum_{j=-\infty}^{\infty}(1-z^{j-1})(1-z^j)z^{1-j}q^{2j^2-j}.\end{aligned}$$

□

Proof of (2.6). We have by (3.1) that

$$(1+z)\left(z,z^{-1};q^2\right)_\infty S_{AG4}(z,q) = \frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{(1+z)\left(z,z^{-1};q^2\right)_n(-1)^nq^{n^2}}{\left(-q;q^2\right)_n\left(q^4;q^4\right)_n}$$

$$= \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(1+z)\left(z,z^{-1};q^2\right)_n (-1)^n q^{n^2} \left(q;q^2\right)_n}{(q^2;q^2)_{2n}}$$
$$= \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{n^2} \left(q;q^2\right)_n \sum_{j=-n}^{n+1} \frac{(-1)^{j+1}(1-q^{4j-2})z^j q^{j(j-3)+2}}{(q^2;q^2)_{n+j} (q^2;q^2)_{n-j+1}}.$$

We note the coefficients of z^{-j} and z^{j+1} are then the same in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{AG4}(z,q)$, so we need only determine the coefficients of z^j for $j \geq 1$. This time we use (3.3). For $j \geq 2$, the coefficient of z^j in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{AG4}(z,q)$ is given by

$$\frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \sum_{n=j-1}^{\infty} \frac{(-1)^{j+n+1} q^{n^2} \left(q;q^2\right)_n (1-q^{4j-2}) q^{j(j-3)+2}}{(q^2;q^2)_{n+j} (q^2;q^2)_{n-j+1}}$$
$$= \frac{\left(q^2;q^2\right)_\infty (1-q^{4j-2}) q^{2j^2-5j+3}}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2jn-2n} \left(q;q^2\right)_{n+j-1}}{(q^2;q^2)_{n+2j-1} (q^2;q^2)_n}$$
$$= \frac{\left(q^2;q^2\right)_\infty \left(q;q^2\right)_{j-1} (1-q^{4j-2}) q^{2j^2-5j+3}}{(q;q^2)_\infty (q^2;q^2)_{2j-1}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2jn-2n} \left(q^{2j-1};q^2\right)_n}{(q^{4j};q^2)_n (q^2;q^2)_n}$$
$$= \frac{\left(q^2;q^2\right)_\infty \left(q;q^2\right)_{j-1} (1-q^{4j-2}) q^{2j^2-5j+3}}{(q;q^2)_\infty (q^2;q^2)_{2j-1}} \sum_{n=0}^{\infty} (-1)^n q^{n^2+2jn-2n} \left(q^{2j-1};q^2\right)_n \beta_n^{**}(q^{4j-4},q^2)$$
$$= \frac{\left(q^2;q^2\right)_\infty \left(q;q^2\right)_{j-1} (1-q^{4j-2}) q^{2j^2-5j+3}}{(q;q^2)_\infty (q^2;q^2)_{2j-1}} \frac{\left(q^{2j-1};q^2\right)_\infty}{(q^{4j-2};q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n^2+2jn-2n} \alpha_n^{**}(q^{4j-4},q^2)$$
$$= q^{2j^2-5j+3}(1+q^{6j-3}).$$

Similarly, the coefficient of z in $(1+z)\left(z,z^{-1};q^2\right)_\infty S_{AG4}(z,q)$ is given by

$$\begin{aligned}
\frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2} \left(q;q^2\right)_n (1-q^2)}{(q^2;q^2)_{n+1} (q^2;q^2)_n} &= \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} \left(q;q^2\right)_n}{(q^4;q^2)_n (q^2;q^2)_n} - \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \\
&= \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n^2} \left(q;q^2\right)_n \beta_n^{**}(1;q^2) - \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \\
&= \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \frac{\left(q;q^2\right)_\infty}{(q^2;q^2)_\infty} (1+q^3) - \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} \\
&= 1+q^3 - \frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty}.
\end{aligned}$$

Along with (3.11), this gives

$$\begin{aligned}
(1+z)\left(z,z^{-1};q^2\right)_\infty S_{AG4}(z,q) &= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} + \sum_{j=1}^{\infty} (z^j + z^{1-j})(1+q^{6j-3}) q^{2j^2-5j+3} \\
&= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} + \sum_{j=-\infty}^{\infty} z^j (1+q^{6j-3}) q^{2j^2-5j+3} \\
&= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} + \sum_{j=-\infty}^{\infty} z^j q^{2j^2-5j+3} + \sum_{j=-\infty}^{\infty} z^j q^{2j^2+j} \\
&= -(1+z)\frac{\left(q^2;q^2\right)_\infty}{(q;q^2)_\infty} + \sum_{j=-\infty}^{\infty} z^{1-j} q^{2j^2+j} + \sum_{j=-\infty}^{\infty} z^j q^{2j^2+j}
\end{aligned}$$

$$= \sum_{j=-\infty}^{\infty} (z^j + z^{1-j} - 1 - z) q^{2j^2+j}$$
$$= \sum_{j=-\infty}^{\infty} (1 - z^{j-1})(1 - z^j) z^{1-j} q^{2j^2+j}.$$

$\square$

4. Dissections for $S_{B2}(z,q)$

To begin, by Bailey's Lemma with $\rho_1 = z$ and $\rho_2 = z^{-1}$ we have that

$$\begin{aligned}(1-z)(1-z^{-1})S_{B2}(z,q) &= \frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}\sum_{n=0}^{\infty}\frac{\left(z,z^{-1};q\right)_n q^{2n}}{(q;q)_n} - \frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}\\ &= \frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}\sum_{n=0}^{\infty}\left(z,z^{-1};q\right)_n q^n \beta_n^{B2} - \frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}\\ &= \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{(3n^2-n)/2}(1+q^{3n})}{(1-zq^n)(1-z^{-1}q^n)}\right) - \frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}.\end{aligned}$$

While the series term is not the generating function for the rank of partitions, it is surprisingly close to it. We recall the rank of a partition is the largest part minus the number of parts. One form of the generating function for the rank of partitions, which is given on page 64 of [28], is

$$R(z,q) = \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right). \tag{4.1}$$

We recall the crank of a partition is the largest part, if there are no ones, and otherwise is the number of parts larger than the number of ones minus the number of ones. One form of the generating function for the crank of partitions, which is given in (7.15) of [16], is

$$C(z,q) = \frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}. \tag{4.2}$$

Lemma 4.1.

$$\frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n-1)/2}(1+q^{3n})}{(1-zq^n)(1-z^{-1}q^n)}\right) = (z+z^{-1}-1)R(z,q) + (1-z)(1-z^{-1}).$$

Proof. To prove this identity, we multiply both sides by $(q;q)_\infty$ and expand $(q;q)_\infty$ into a series by Euler's pentagonal number theorem. We then have

$$\begin{aligned}&(q;q)_\infty\left((z+z^{-1}-1)R(z,q)+(1-z)(1-z^{-1})\right)\\ &= z+z^{-1}-1+(1-z)(1-z^{-1})\,(q;q)_\infty + \sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}(1+q^n)(z+z^{-1}-1)}{(1-zq^n)(1-z^{-1}q^n)}\\ &= 1+\sum_{n=1}^{\infty}(1-z)(1-z^{-1})(-1)^n q^{n(3n-1)/2}(1+q^n)\left(\frac{q^n(z+z^{-1}-1)}{(1-zq^n)(1-z^{-1}q^n)}+1\right)\\ &= 1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n-1)/2}(1+q^n)(1-q^n+q^{2n})}{(1-zq^n)(1-z^{-1}q^n)}\\ &= 1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n-1)/2}(1+q^{3n})}{(1-zq^n)(1-z^{-1}q^n)}.\end{aligned}$$

This proves the lemma. $\square$

With Lemma 4.1 we now have

$$S_{B2}(z,q) = \frac{(z+z^{-1}-1)R(z,q) - C(z,q)}{(1-z)(1-z^{-1})} + 1. \tag{4.3}$$

Using the rank difference formulas from [7], we can deduce the following dissections for the rank function. Theorem 4 of [7] gives the following dissection for $R(\zeta_5,q)$, which can also be found as Entry 2.1.2 in [4],

$$\begin{aligned} R(\zeta_5,q) = {} & \frac{\left(q^{25};q^{25}\right)_\infty \left[q^{10};q^{25}\right]_\infty}{\left[q^5;q^{25}\right]_\infty^2} + q^5 \frac{\zeta_5+\zeta_5^{-1}-2}{(q^{25};q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{75n(n+1)/2}}{1-q^{25n+5}} + q\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^5;q^{25}\right]_\infty} \\ & + q^2(\zeta_5+\zeta_5^{-1}) \frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{25}\right]_\infty} - q^3(\zeta_5+\zeta_5^{-1}) \frac{\left(q^{25};q^{25}\right)_\infty \left[q^5;q^{25}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty^2} \\ & - q^8 \frac{2\zeta_5+2\zeta_5^{-1}+1}{(q^{25};q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{75n(n+1)/2}}{1-q^{25n+10}}. \end{aligned} \tag{4.4}$$

Similarly Theorem 5 of [7] gives the following dissection for $R(\zeta_7,q)$, which is also Entry 2.1.5 of [4].

$$\begin{aligned} & R(\zeta_7,q) \\ & = (1-\zeta_7)(1-\zeta_7^6) + (-1+\zeta_7+\zeta_7^6) \frac{\left(q^{49};q^{49}\right)_\infty \left[q^{21};q^{49}\right]_\infty}{\left[q^7,q^{14};q^{49}\right]_\infty} \\ & \quad + (2-\zeta_7-\zeta_7^6) q^7 \frac{1}{(q^{49};q^{49})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{147n(n+1)/2}}{1-q^{49n+7}} + q\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^7;q^{49}\right]_\infty} \\ & \quad + (\zeta_7+\zeta_7^6) q^2 \frac{\left(q^{49};q^{49}\right)_\infty \left[q^{14};q^{49}\right]_\infty}{\left[q^7,q^{21};q^{49}\right]_\infty} + (\zeta_7-\zeta_7^2-\zeta_7^5+\zeta_7^6) q^{16} \frac{1}{(q^{49};q^{49})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{147n(n+1)/2}}{1-q^{49n+21}} \\ & \quad + (1+\zeta_7^2+\zeta_7^5) q^3 \frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{14};q^{49}\right]_\infty} - (\zeta_7^2+\zeta_7^5) q^4 \frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{21};q^{49}\right]_\infty} \\ & \quad + (1+\zeta_7+2\zeta_7^2+2\zeta_7^5+\zeta_7^6) q^{13} \frac{1}{(q^{49};q^{49})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{147n(n+1)/2}}{1-q^{49n+14}} \\ & \quad + (\zeta_7+\zeta_7^2+\zeta_7^5+\zeta_7^6) q^6 \frac{\left(q^{49};q^{49}\right)_\infty \left[q^7;q^{49}\right]_\infty}{\left[q^{14},q^{21};q^{49}\right]_\infty}. \end{aligned} \tag{4.5}$$

Next by (3.8) of [16]

$$\begin{aligned} \frac{(q;q)_\infty}{\left(\zeta_5 q, \zeta_5^{-1} q; q\right)_\infty} = \left(q^{25};q^{25}\right)_\infty \Bigg(& \frac{\left[q^{10};q^{25}\right]_\infty}{\left[q^5;q^{25}\right]_\infty^2} + (\zeta_5+\zeta_5^4-1) q \frac{1}{\left[q^5;q^{25}\right]_\infty} - (\zeta_5+\zeta_5^4+1) q^2 \frac{1}{\left[q^{10};q^{25}\right]_\infty} \\ & -(\zeta_5+\zeta_5^4) q^3 \frac{\left[q^5;q^{25}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty^2} \Bigg). \end{aligned} \tag{4.6}$$

Also by Theorem 5.1 of [16] we have

$$\begin{aligned} \frac{(q;q)_\infty}{\left(\zeta_7 q, \zeta_7^{-1} q; q\right)_\infty} = \left(q^{49};q^{49}\right)_\infty \Bigg(& \frac{\left[q^{21};q^{49}\right]_\infty}{\left[q^7,q^{14};q^{49}\right]_\infty} + (\zeta_7+\zeta_7^6-1) q \frac{1}{\left[q^7;q^{49}\right]_\infty} + (\zeta_7^2+\zeta_7^5) q^2 \frac{\left[q^{14};q^{49}\right]_\infty}{\left[q^7,q^{21};q^{49}\right]_\infty} \\ & -(\zeta_7+\zeta_7^2+\zeta_7^5+\zeta_7^6) q^3 \frac{1}{\left[q^{14};q^{49}\right]_\infty} - (\zeta_7+\zeta_7^6) q^4 \frac{1}{\left[q^{21};q^{49}\right]_\infty} \\ & -(\zeta_7^2+\zeta_7^5+1) q^6 \frac{\left[q^7;q^{49}\right]_\infty}{\left[q^{14},q^{21};q^{49}\right]_\infty} \Bigg). \end{aligned} \tag{4.7}$$

We then find (2.7) of Theorem 2.3 follows by (4.3), (4.4), and (4.6) and (2.8) of Theorem 2.3 follows by (4.3), (4.5), and (4.7).

5. Dissections of $S_{F3}(z,q)$

By Bailey's Lemma with $\rho_1 = z$ and $\rho_2 = z^{-1}$ we have that

$$\begin{aligned}(1-z)(1-z^{-1})S_{F3}(z,q) &= \frac{(q;q)_\infty}{(zq^2,z^{-1}q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n q^n}{(q;q)_{2n}} - \frac{(q;q)_\infty}{(zq^2,z^{-1}q^2;q^2)_\infty}\\ &= \frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})q^n(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) - \frac{(q;q)_\infty}{\left(zq^2,z^{-1}q^2;q\right)_\infty}.\end{aligned}$$

We first find the dissections for the product term and then proceed with the series term. When $z=\zeta_7$ we use the theory of modular functions, both for the product term and the series term. We recall some facts about modular functions as in [24] and use the notation in [9] and [25]. The generalized eta function is defined by

$$\eta_{\delta,g}(\tau) = q^{P(g/\delta)\delta/2}\prod_{\substack{n>0\\ n\equiv g \pmod d}}(1-q^n)\prod_{\substack{n>0\\ n\equiv -g \pmod d}}(1-q^n),$$

where $q=e^{2\pi i\tau}$ and $P(t)=\{t\}^2-\{t\}+\frac{1}{6}$. So $\eta_{\delta,0}(\tau)=q^{\delta/12}\left(q^\delta;q^\delta\right)_\infty^2$ and $\eta_{\delta,g}(\tau)=q^{P(g/\delta)\delta/2}\left[q^g;q^\delta\right]_\infty$ for $0<g<\delta$. We use Theorem 3 of [25] to determine when a quotient of $\eta_{\delta,g}(\tau)$ is a modular function with respect to a congruence subgroup $\Gamma_1(N)$ and use Theorem 4 of [25] to determine the order at the cusps. Suppose f is a modular function with respect to the congruence subgroup Γ of $\Gamma_0(1)$. For $A\in\Gamma_0(1)$ we have a cusp given by $\zeta=A^{-1}\infty$. The width of the cusp $W:=W(\Gamma,\zeta)$ is given by

$$W(\Gamma,\zeta)=\min\{k>0:\pm A^{-1}T^kA\in\Gamma\},$$

where T is the translation matrix

$$T=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

If

$$f(A^{-1}\tau)=\sum_{m=m_0}^{\infty}b_m q^{m/W}$$

and $b_{m_0}\neq 0$, then we say m_0 is the order of f at ζ with respect to Γ and we denote this value by $Ord_\Gamma(f;\zeta)$. By $ord(f;\zeta)$ we mean the invariant order of f at ζ given by

$$ord(f;\zeta)=\frac{Ord_\Gamma(f;\zeta)}{W}.$$

For z in the upper half plane $\mathcal{H}$, we write $ord(f;z)$ for the order of f at z as an analytic function in z. We define the order of f at z with respect to Γ by

$$Ord_\Gamma(f;z)=\frac{ord(f;z)}{m},$$

where m is the order of z as a fixed point of Γ.

The valence formula for modular functions is as follows. Suppose a subset $\mathcal{F}$ of $\mathcal{H}\cup\{\infty\}\cup\mathbb{Q}$ is a fundamental region for the action of Γ along with a complete set of inequivalent cusps, if f is not the zero function then

$$\sum_{z\in\mathcal{F}}Ord_\Gamma(f;z)=0. \tag{5.1}$$

We can verify an identity between sums of generalized eta quotients as follows. Suppose we are to show

$$a_1f_1+a_2f_2+\cdots+a_kf_k=a_{k+1}f_{k+1}+a_{k+2}f_{k+2}+\cdots+a_{k+m}f_{k+m},$$

where each $a_i\in\mathbb{C}$ and each f_i is of the form

$$f_i=\prod_{j=1}^{n}\eta_{\delta_j,g_j}(\tau)^{r_j}.$$

We verify each f_i is a modular function with respect to a common $\Gamma_1(N)$, so that $f = a_1 f_1 + \cdots + a_k f_k - a_{k+1} f_{k+1} - \cdots - a_{k+m} f_{k+m}$ is a modular function with respect to $\Gamma_1(N)$. Although f may have zeros at points other than the cusps, the poles must occur only at the cusps. At each cusp ζ, not equivalent to ∞, we compute a lower bound for $Ord_\Gamma(f;\zeta)$ by taking the minimum of the $Ord_\Gamma(f_i,\zeta)$, we call this lower bound B_ζ. We then use the q-expansion of f to find that $Ord_\Gamma(f;\infty)$ is larger than $-\sum_{\zeta\in\mathcal{C}'} B_\zeta$, where $\mathcal{C}'$ is a set of cusps with a representative of each cusp not equivalent to ∞. By the valence formula we have $f \equiv 0$ since $\sum_{z\in\mathcal{F}} Ord_\Gamma(f;z) > 0$.

Proposition 5.1.

$$\frac{\left(q,q^2;q^2\right)_\infty}{\left(\zeta_3 q^2,\zeta_3^{-1}q^2;q^2\right)_\infty} = \frac{\left(q^9;q^9\right)_\infty^4}{\left(q^{18};q^{18}\right)_\infty^2\left(q^3;q^3\right)_\infty} - q\frac{\left(q^{18};q^{18}\right)_\infty\left(q^9;q^9\right)_\infty}{\left(q^6;q^6\right)_\infty} - 2q^2\frac{\left(q^{18};q^{18}\right)_\infty^4\left(q^3;q^3\right)_\infty}{\left(q^9;q^9\right)_\infty^2\left(q^6;q^6\right)_\infty^2}, \tag{5.2}$$

$$\begin{aligned}\frac{\left(q,q^2;q^2\right)_\infty}{\left(\zeta_5 q^2,\zeta_5^4 q^2;q^2\right)_\infty} &= \frac{\left(q^{25};q^{25}\right)_\infty\left[q^{15};q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty} - q\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{50}\right]_\infty} + (\zeta_5+\zeta_5^4)q^2\frac{\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{25}\right]_\infty}{\left[q^5;q^{25}\right]_\infty\left[q^{20};q^{50}\right]_\infty}\\
&\quad - q^2\frac{\left(q^{25};q^{25}\right)_\infty\left[q^5;q^{25}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty\left[q^{10};q^{50}\right]_\infty} - (\zeta_5+\zeta_5^4)q^3\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{20};q^{50}\right]_\infty}\\
&\quad - (\zeta_5+\zeta_5^4)q^4\frac{\left(q^{25};q^{25}\right)_\infty\left[q^5;q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty},\end{aligned} \tag{5.3}$$

$$\begin{aligned}\frac{\left(q,q^2;q^2\right)_\infty}{\left(\zeta_7 q^2,\zeta_7^{-1}q^2;q^2\right)_\infty} &= \frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{14};q^{98}\right]_\infty} - (\zeta_7^2+\zeta_7^5)q\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{35};q^{98}\right]_\infty}{\left[q^{21};q^{49}\right]_\infty}\\
&\quad + (1+\zeta_7^2+\zeta_7^5)q\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{21};q^{98}\right]_\infty}{\left[q^7;q^{49}\right]_\infty} - 2q\frac{\left(q^{98};q^{98}\right)_\infty\left[q^{35};q^{98}\right]_\infty}{\left(q^{49};q^{98}\right)_\infty\left[q^{14};q^{98}\right]_\infty}\\
&\quad - (1-\zeta_7-\zeta_7^6)q^2\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{14};q^{49}\right]_\infty}{\left[q^7;q^{49}\right]_\infty\left[q^{28};q^{98}\right]_\infty} - (\zeta_7+\zeta_7^6)q^3\frac{\left(q^{49};q^{49}\right)_\infty\left[q^7;q^{49}\right]_\infty}{\left[q^{21};q^{49}\right]_\infty\left[q^{14};q^{98}\right]_\infty}\\
&\quad + (1+\zeta_7^2+\zeta_7^5)q^4\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{28};q^{98}\right]_\infty} - (\zeta_7+\zeta_7^2+\zeta_7^5+\zeta_7^6)q^5\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{21};q^{49}\right]_\infty}{\left[q^{14};q^{49}\right]_\infty\left[q^{42};q^{98}\right]_\infty}\\
&\quad - (\zeta_7^2+\zeta_7^5)q^6\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{42};q^{98}\right]_\infty}.\end{aligned} \tag{5.4}$$

Proof. Equation (5.2) follows from Theorem 2.11 of [14] by replacing q by $-q$ and simplifying the products. Similarly (5.3) follows from Theorem 2.12 of [14] with q replaced by $-q$.

We recognize the left hand side of (5.4) as $\left(q;q^2\right)_\infty C(\zeta_7,q^2)$, where $C(z,q)$ is defined in (4.2). For $C(\zeta_7,q^2)$, we use (4.7) with q replaced by q^2. If we divide both sides by $\frac{\left(q;q^2\right)_\infty\left(q^{98};q^{98}\right)_\infty\left[q^{42};q^{98}\right]_\infty}{\left[q^{14},q^{28};q^{98}\right]_\infty}$, we find that (5.4) is equivalent to

$$\begin{aligned}&1 + (\zeta_7^6+\zeta_7-1)\frac{\eta_{98,28}(\tau)}{\eta_{98,42}(\tau)} + (\zeta_7^5+\zeta_7^2)\frac{\eta_{98,28}(\tau)^2}{\eta_{98,42}(\tau)^2} + (\zeta_7^4+\zeta_7^3+1)\frac{\eta_{98,14}(\tau)}{\eta_{98,42}(\tau)} - (\zeta_7^6+\zeta_7)\frac{\eta_{98,14}(\tau)\,\eta_{98,28}(\tau)}{\eta_{98,42}(\tau)^2}\\
&\quad - (\zeta_7^5+\zeta_7^2+1)\frac{\eta_{98,14}(\tau)^2}{\eta_{98,42}(\tau)^2}\\
&= \frac{\eta_{98,28}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{98,42}(\tau)} - (\zeta_7^5+\zeta_7^2)\frac{\eta_{98,14}(\tau)\,\eta_{98,28}(\tau)\,\eta_{98,35}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{49,21}(\tau)\,\eta_{98,42}(\tau)}\\
&\quad + (\zeta_7^5+\zeta_7^2+1)\frac{\eta_{98,14}(\tau)\,\eta_{98,21}(\tau)\,\eta_{98,28}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{49,7}(\tau)\,\eta_{98,42}(\tau)} - 2\frac{\eta_{98,28}(\tau)\,\eta_{98,35}(\tau)}{\eta_{2,1}(\tau)^{1/2}\,\eta_{98,42}(\tau)\,\eta_{98,49}(\tau)^{1/2}}\\
&\quad - (-\zeta_7^6-\zeta_7+1)\frac{\eta_{49,14}(\tau)\,\eta_{98,14}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{49,7}(\tau)\,\eta_{98,42}(\tau)} - (\zeta_7^6+\zeta_7)\frac{\eta_{49,7}(\tau)\,\eta_{98,28}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{49,21}(\tau)\,\eta_{98,42}(\tau)}\end{aligned}$$

$$+ (\zeta_7^5+\zeta_7^2+1)\frac{\eta_{98,14}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{98,42}(\tau)} - (\zeta_7^6+\zeta_7^5+\zeta_7^2+\zeta_7)\frac{\eta_{49,21}(\tau)\,\eta_{98,14}(\tau)\,\eta_{98,28}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{49,14}(\tau)\,\eta_{98,42}(\tau)^2}$$
$$- (\zeta_7^5+\zeta_7^2)\frac{\eta_{98,14}(\tau)\,\eta_{98,28}(\tau)\,\eta_{98,49}(\tau)^{1/2}}{\eta_{2,1}(\tau)^{1/2}\,\eta_{98,42}(\tau)^2}. \tag{5.5}$$

However, by Theorem 3 of [25] each individual term of (5.5) is a modular function with respect to $\Gamma_1(98)$. Using Theorem 4 of [25] to compute the orders at the cusps, as explained previously, we find to prove (5.5) that we need only check this identity in the q-series expansion past q^{210}. This we do with Maple and so (5.4) is true.

□

Proposition 5.2.

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-\zeta_3)(1-\zeta_3^{-1})q^n(1+q^{2n})}{(1-\zeta_3q^{2n})(1-\zeta_3^{-1}q^{2n})}\right)$$
$$= \frac{(q^9;q^9)_\infty^4}{(q^3;q^3)_\infty\,(q^{18};q^{18})_\infty^2} + 2q\frac{(q^9;q^9)_\infty\,(q^{18};q^{18})_\infty}{(q^6;q^6)_\infty} + q^2\frac{(q^3;q^3)_\infty\,(q^{18};q^{18})_\infty^4}{(q^6;q^6)_\infty^2\,(q^9;q^9)_\infty^2}, \tag{5.6}$$

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-\zeta_5)(1-\zeta_5^{-1})q^n(1+q^{2n})}{(1-\zeta_5q^{2n})(1-\zeta_5q^{2n})}\right)$$
$$= \frac{(q^{25};q^{25})_\infty\,[q^{15};q^{50}]_\infty}{[q^5;q^{25}]_\infty} + (1-\zeta_5-\zeta_5^4)q\frac{(q^{25};q^{25})_\infty}{[q^{10};q^{50}]_\infty} + q^2\frac{(q^{50};q^{50})_\infty\,[q^{15};q^{50}]_\infty}{(q^{25};q^{50})_\infty\,[q^{10};q^{50}]_\infty}$$
$$- (\zeta_5+\zeta_5^4)q^7\frac{(q^{50};q^{50})_\infty\,[q^5;q^{50}]_\infty}{(q^{25};q^{50})_\infty\,[q^{20};q^{50}]_\infty} + (1+\zeta_5+\zeta_5^4)q^3\frac{(q^{25};q^{25})_\infty}{[q^{20};q^{50}]_\infty} - (\zeta_5+\zeta_5^4)q^4\frac{(q^{25};q^{25})_\infty\,[q^5;q^{50}]_\infty}{[q^{10};q^{25}]_\infty}, \tag{5.7}$$

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-\zeta_7)(1-\zeta_7^{-1})q^n(1+q^{2n})}{(1-\zeta_7q^{2n})(1-\zeta_7q^{2n})}\right)$$
$$= \frac{(q^{49};q^{49})_\infty}{[q^{14};q^{98}]_\infty} + q\frac{(q^{98};q^{98})_\infty\,[q^{35};q^{98}]_\infty}{(q^{49};q^{98})_\infty\,[q^{14};q^{98}]_\infty} - (\zeta_7+\zeta_7^6)q\frac{(q^{14};q^{14})_\infty}{(q^7;q^{14})_\infty} - (\zeta_7^2+\zeta_7^5)q^8\frac{(q^{98};q^{98})_\infty\,[q^{21};q^{98}]_\infty}{(q^{49};q^{98})_\infty\,[q^{28};q^{98}]_\infty}$$
$$+ q^2\frac{(q^{49};q^{49})_\infty\,[q^{14};q^{49}]_\infty}{[q^7;q^{49}]_\infty\,[q^{28};q^{98}]_\infty} - (\zeta_7^2+\zeta_7^5)q^3\frac{(q^{49};q^{49})_\infty\,[q^7;q^{49}]_\infty}{[q^{21};q^{49}]_\infty\,[q^{14};q^{98}]_\infty} + (1+\zeta_7^2+\zeta_7^5)q^4\frac{(q^{49};q^{49})_\infty}{[q^{28};q^{98}]_\infty}$$
$$+ (1+\zeta_7^2+\zeta_7^5)q^5\frac{(q^{49};q^{49})_\infty\,[q^{21};q^{49}]_\infty}{[q^{14};q^{49}]_\infty\,[q^{42};q^{98}]_\infty} - (\zeta_7^2+\zeta_7^5)q^6\frac{(q^{49};q^{49})_\infty}{[q^{42};q^{98}]_\infty}. \tag{5.8}$$

We see that (2.9) follows by subtracting (5.2) from (5.6) and dividing by $(1-\zeta_3)(1-\zeta_3^{-1})$, (2.10) follows by subtracting (5.3) from (5.7) and dividing by $(1-\zeta_5)(1-\zeta_5^{-1})$, and (2.11) follows by subtracting (5.4) from (5.8) and dividing by $(1-\zeta_7)(1-\zeta_7^{-1})$. While we can prove (5.6) with elementary rearrangements, the proofs of (5.7) and (5.8) will require the following identities. We use equation (17.1) from [9, page 303], which in our notation is

$$\frac{q}{ab}\frac{(q^2;q^2)_\infty^4\,[a,b,ab;q^2]_\infty}{(q;q)_\infty^2\,[aq,bq,abq;q^2]_\infty} = \sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\left(\frac{1}{a^nb^n}-\frac{1}{a^n}-\frac{1}{b^n}+a^n+b^n-a^nb^n\right). \tag{5.9}$$

We also use Theorem 1 of [6] with $b=a$ and $c=q^{1/2}$,

$$\frac{(q;q)_\infty^2\,[a^2,aq^{1/2},aq^{1/2};q]_\infty}{[a,a,q^{1/2},a^2q^{1/2};q]_\infty} = 1+2\sum_{k=0}^{\infty}\frac{aq^k}{1-aq^k} - 2\sum_{k=1}^{\infty}\frac{q^k/a}{1-q^k/a} - \sum_{k=0}^{\infty}\frac{a^2q^{k+1/2}}{1-a^2q^{k+1/2}} + \sum_{k=1}^{\infty}\frac{q^{k-1/2}/a^2}{1-q^{k-1/2}/a^2}. \tag{5.10}$$

Lastly, we will use the following dissection formula for certain quotients of theta functions.

Lemma 5.3. *Let M be a positive integer and $|q^{2M}| < |z| < 1$, then*

$$\frac{\left(q^{2M};q^{2M}\right)_\infty^2\left[zq^M;q^{2M}\right]_\infty}{\left(q^M;q^{2M}\right)_\infty^2\left[z;q^{2M}\right]_\infty} = \frac{\left(q^{2M^2};q^{2M^2}\right)_\infty^2}{\left[z^M;q^{2M^2}\right]_\infty}\sum_{k\in A} z^k\frac{\left[z^Mq^{M(2k+1)};q^{2M^2}\right]_\infty}{\left[q^{M(2k+1)};q^{2M^2}\right]_\infty},$$

where A is any full set of residues modulo N (such as $A = \{0,1,2,\ldots,M-1\}$).

Proof. We recall a specialization of Ramanujan's ${}_1\Psi_1$ formula gives

$$\frac{(q;q)_\infty^2\,[xy;q]_\infty}{[x,y;q]_\infty} = \sum_{n=-\infty}^{\infty}\frac{x^n}{1-yq^n}$$

for $|q| < |x| < 1$. We let $q \mapsto q^{2M}$, $x = z$, and $y = q^M$ to find that

$$\begin{aligned}
\frac{\left(q^{2M};q^{2M}\right)_\infty^2\left[zq^M;q^{2M}\right]_\infty}{\left(q^M;q^{2M}\right)_\infty^2\left[z;q^{2M}\right]_\infty} &= \sum_{n=-\infty}^{\infty}\frac{z^n}{1-q^{2Mn+M}}\\
&= \sum_{k\in A}\sum_{n=-\infty}^{\infty}\frac{z^{Mn+k}}{1-q^{2M^2n+2Mk+M}}\\
&= \sum_{k\in A} z^k\sum_{n=-\infty}^{\infty}\frac{z^{Mn}}{1-q^{2M^2n+2Mk+M}}\\
&= \sum_{k\in A} z^k\frac{\left[z^Mq^{M(2k+1)};q^{2M^2}\right]_\infty\left(q^{2M^2};q^{2M^2}\right)_\infty^2}{\left[z^M,q^{M(2k+1)};q^{2M^2}\right]_\infty}.
\end{aligned}$$

□

In particular, we can set $z = \pm q^a$ for $1 \le a < 2M$. Similar dissection formulas for certain quotients of theta functions follow from both the quintuple product identity and Theorem 2.1 of [10].

Proof of (5.6). We have that

$$\begin{aligned}
1+\sum_{n=1}^{\infty}\frac{(1-\zeta_3)(1-\zeta_3^{-1})q^n(1+q^{2n})}{(1-\zeta_3q^{2n})(1-\zeta_3^{-1}q^{2n})} &= 1+3\sum_{n=1}^{\infty}\frac{q^n(1+q^{2n})(1-q^{2n})}{(1-q^{6n})}\\
&= 1+3\sum_{n=1}^{\infty}\frac{q^n-q^{5n}}{1-q^{6n}}\\
&= 1+3\sum_{n=1}^{\infty}q^nE_1(n:6)\\
&= \frac{\left(q^2;q^2\right)_\infty^6\left(q^3;q^3\right)_\infty}{(q;q)_\infty^3\left(q^6;q^6\right)_\infty^2},
\end{aligned}$$

where

$$E_r(N;m) = \sum_{d|N,d\equiv r \pmod m} 1 - \sum_{d|N,d\equiv -r \pmod m} 1,$$

by (32.42) of [13]. By Gauss and the Jacobi Triple Product Identity

$$\begin{aligned}
\frac{\left(q^2;q^2\right)_\infty^2}{(q;q)_\infty} &= \frac{1}{2}\sum_{n=-\infty}^{\infty}q^{n(n+1)/2}\\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty}q^{(9n^2+3n)/2}+\frac{q}{2}\sum_{n=-\infty}^{\infty}q^{(9n^2+9n)/2}+\frac{q^3}{2}\sum_{n=-\infty}^{\infty}q^{(9n^2+15n)/2}\\
&= \frac{1}{2}\left(-q^3,-q^6,q^9;q^9\right)_\infty+\frac{q}{2}\left(-1,-q^9,q^9;q^9\right)_\infty+\frac{q^3}{2}\left(-q^{-3},-q^{12},q^9;q^9\right)_\infty
\end{aligned}$$

$$= \left(-q^3,-q^6,q^9;q^9\right)_\infty + q\left(-q^9,-q^9,q^9;q^9\right)_\infty.$$

Thus

$$\begin{aligned}\frac{\left(q^2;q^2\right)_\infty^4}{(q;q)_\infty^2} &= \left(-q^3,-q^6,q^9;q^9\right)_\infty^2 + 2q\left(-q^3,-q^6,-q^9,-q^9,q^9,q^9;q^9\right)_\infty + q^2\left(-q^9,-q^9,q^9;q^9\right)_\infty^2 \\ &= \left(-q^3,-q^6,q^9;q^9\right)_\infty^2 + 2q\left(-q^3;q^3\right)_\infty\left(q^9;q^9\right)_\infty\left(q^{18};q^{18}\right)_\infty + q^2\left(-q^9;q^9\right)_\infty^2\left(q^{18};q^{18}\right)_\infty^2.\end{aligned}$$

And so

$$\begin{aligned}&\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-\zeta_3)(1-\zeta_3^{-1})q^n(1+q^{2n})}{(1-\zeta_3q^{2n})(1-\zeta_3^{-1}q^{2n})}\right) \\ &= \frac{\left(q^2;q^2\right)_\infty^4\left(q^3;q^3\right)_\infty}{(q;q)_\infty^2\left(q^6;q^6\right)_\infty^2} \\ &= \frac{\left(q^3;q^3\right)_\infty\left(-q^3,-q^6,q^9;q^9\right)_\infty^2}{\left(q^6;q^6\right)_\infty^2} + 2q\frac{\left(q^9;q^9\right)_\infty\left(q^{18};q^{18}\right)_\infty}{\left(q^6;q^6\right)_\infty} + q^2\frac{\left(q^3;q^3\right)_\infty\left(-q^9;q^9\right)_\infty^2\left(q^{18};q^{18}\right)_\infty^2}{\left(q^6;q^6\right)_\infty^2} \\ &= \frac{\left(q^9;q^9\right)_\infty^4}{\left(q^3;q^3\right)_\infty\left(q^{18};q^{18}\right)_\infty^2} + 2q\frac{\left(q^9;q^9\right)_\infty\left(q^{18};q^{18}\right)_\infty}{\left(q^6;q^6\right)_\infty} + q^2\frac{\left(q^3;q^3\right)_\infty\left(q^{18};q^{18}\right)_\infty^4}{\left(q^6;q^6\right)_\infty^2\left(q^9;q^9\right)_\infty^2}.\end{aligned}$$

□

Proof of (5.7) . To begin, we have

$$\begin{aligned}&1+(1-\zeta_5)(1-\zeta_5^{-1})\sum_{n=1}^{\infty}\frac{q^n(1+q^{2n})}{(1-\zeta_5q^{2n})(1-\zeta_5q^{2n})} \\ &= 1+(1-\zeta_5)(1-\zeta_5^{-1})\sum_{n=1}^{\infty}\frac{q^n(1+q^{2n})(1-q^{2n})(1-\zeta_5^2q^{2n})(1-\zeta_5^3q^{2n})}{1-q^{10n}} \\ &= 1+(2-\zeta_5-\zeta_5^4)\sum_{n=1}^{\infty}\frac{q^n-q^{9n}}{1-q^{10n}} + (1+2\zeta_5+2\zeta_5^4)\sum_{n=1}^{\infty}\frac{q^{3n}-q^{7n}}{1-q^{10n}}.\end{aligned}$$

We claim that

$$\begin{aligned}&\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{q^n-q^{9n}}{1-q^{10n}}+\sum_{n=1}^{\infty}\frac{q^{3n}-q^{7n}}{1-q^{10n}}\right) \\ &= \frac{\left(q^{10};q^{10}\right)_\infty\left[q^4;q^{10}\right]_\infty}{\left(q^5;q^{10}\right)_\infty\left[q;q^{10}\right]_\infty} \\ &= \frac{\left(q^{25};q^{25}\right)_\infty\left[q^{15};q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty} + q\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{50}\right]_\infty} + q^2\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{15};q^{50}\right]_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{10};q^{50}\right]_\infty} + q^3\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{20};q^{50}\right]_\infty}.\end{aligned} \tag{5.11}$$

We note the second identity of (5.11) is just an application of Lemma 5.3, with $M=5$ and $z=q$ and simplifying the resulting products. So we need to verify the first identity. We have

$$\begin{aligned}&1+2\sum_{n=1}^{\infty}\frac{q^n-q^{9n}}{1-q^{10n}}+\sum_{n=1}^{\infty}\frac{q^{3n}-q^{7n}}{1-q^{10n}} \\ &= 1+2\sum_{n=1}^{\infty}\frac{q^{10n+1}}{1-q^{10n+1}} - 2\sum_{n=0}^{\infty}\frac{q^{10n-1}}{1-q^{10n-1}} - \sum_{n=0}^{\infty}\frac{q^{10n+7}}{1-q^{10n+7}} + \sum_{n=1}^{\infty}\frac{q^{10n-7}}{1-q^{10n-7}} \\ &= \frac{\left(q^{10};q^{10}\right)_\infty^2\left[q^2,q^6,q^6;q^{10}\right]_\infty}{\left[q,q,q^5,q^7;q^{10}\right]_\infty},\end{aligned}$$

where we have used (5.10) with $q \mapsto q^{10}$ and $a = q$. Thus

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{q^n-q^{9n}}{1-q^{10n}}+\sum_{n=1}^{\infty}\frac{q^{3n}-q^{7n}}{1-q^{10n}}\right)=\frac{\left(q;q^2\right)_\infty\left(q^{10};q^{10}\right)_\infty^2\left[q^2,q^6,q^6;q^{10}\right]_\infty}{\left(q^2;q^2\right)_\infty\left[q,q,q^5,q^7;q^{10}\right]_\infty}$$
$$=\frac{\left(q^{10};q^{10}\right)_\infty\left[q^4;q^{10}\right]_\infty}{\left(q^5;q^{10}\right)_\infty\left[q;q^{10}\right]_\infty}.$$

Next we claim that

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(-\sum_{n=1}^{\infty}\frac{q^n-q^{9n}}{1-q^{10n}}+2\sum_{n=1}^{\infty}\frac{q^{3n}-q^{7n}}{1-q^{10n}}\right)$$
$$=-q\frac{\left(q^{10};q^{10}\right)_\infty\left[q^2;q^{10}\right]_\infty}{\left(q^5;q^{10}\right)_\infty\left[q^3;q^{10}\right]_\infty}$$
$$=-q\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{50}\right]_\infty}-q^7\frac{\left(q^{50};q^{50}\right)_\infty\left[q^5;q^{50}\right]_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{20};q^{50}\right]_\infty}+q^3\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{20};q^{50}\right]_\infty}-q^4\frac{\left(q^{25};q^{25}\right)_\infty\left[q^5;q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}. \tag{5.12}$$

We note the second identity of (5.11) follows by Lemma 5.3 with $M = 5$ and $z = q^3$. For the first identity, we apply (5.9) with $q \mapsto q^5$ and $a = b = q^2$ to get

$$\sum_{n=1}^{\infty}\frac{q^n-q^{9n}}{1-q^{10n}}-2\sum_{n=1}^{\infty}\frac{q^{3n}-q^{7n}}{1-q^{10n}}=q\frac{\left(q^{10};q^{10}\right)_\infty^4\left[q^2,q^2,q^4;q^{10}\right]_\infty}{\left(q^5;q^5\right)_\infty^2\left[q^7,q^7,q^9;q^{10}\right]_\infty}.$$

Thus

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(-2\sum_{n=1}^{\infty}\frac{q^n-q^{9n}}{1-q^{10n}}+2\sum_{n=1}^{\infty}\frac{q^{3n}-q^{7n}}{1-q^{10n}}\right)=-q\frac{\left(q;q^2\right)_\infty\left(q^{10};q^{10}\right)_\infty^4\left[q^2,q^2,q^4;q^{10}\right]_\infty}{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^2\left[q^7,q^7,q^9;q^{10}\right]_\infty}$$
$$=-q\frac{\left(q^{10};q^{10}\right)_\infty\left[q^2;q^{10}\right]_\infty}{\left(q^5;q^{10}\right)_\infty\left[q^3;q^{10}\right]_\infty}.$$

Equation (5.7) now follows from (5.11) and (5.12). $\square$

Proof of (5.8). We begin with

$$1+(1-\zeta_7)(1-\zeta_7^{-1})\sum_{n=1}^{\infty}\frac{q^n(1+q^{2n})}{(1-\zeta_7q^{2n})(1-\zeta_7q^{2n})}$$
$$=1+(1-\zeta_7)(1-\zeta_7^{-1})\sum_{n=1}^{\infty}\frac{q^n(1+q^{2n})(1-q^{2n})(1-\zeta_7^2q^{2n})(1-\zeta_7^3q^{2n})(1-\zeta_7^4q^{2n})(1-\zeta_7^5q^{2n})}{1-q^{14n}}$$
$$=1+(2-\zeta_7-\zeta_7^6)\sum_{n=1}^{\infty}\frac{q^n-q^{13n}}{1-q^{14n}}+(\zeta_7-\zeta_7^2-\zeta_7^5+\zeta_7^6)\sum_{n=1}^{\infty}\frac{q^{3n}-q^{11n}}{1-q^{14n}}$$
$$+(1+\zeta_7+2\zeta_7^2+2\zeta_7^5+\zeta_5^6)\sum_{n=1}^{\infty}\frac{q^{5n}-q^{9n}}{1-q^{14n}}.$$

We claim that

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{2q^n+q^{5n}-q^{9n}-2q^{13n}}{1-q^{14n}}\right)$$
$$=\frac{\left(q^{14};q^{14}\right)_\infty\left[q^3,q^6;q^{14}\right]_\infty}{\left(q^7;q^{14}\right)_\infty\left[q,q^4;q^{14}\right]_\infty}$$
$$=\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{14};q^{98}\right]_\infty}+q\frac{\left(q^{98};q^{98}\right)_\infty\left[q^{35};q^{98}\right]_\infty}{\left(q^{49};q^{98}\right)_\infty\left[q^{14};q^{98}\right]_\infty}+q^2\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{14};q^{49}\right]_\infty}{\left[q^7;q^{49}\right]_\infty\left[q^{28};q^{98}\right]_\infty}+q^4\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{28};q^{98}\right]_\infty}$$
$$+q^5\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{21};q^{49}\right]_\infty}{\left[q^{14};q^{49}\right]_\infty\left[q^{42};q^{98}\right]_\infty}. \tag{5.13}$$

For this we apply (5.10) with $q \mapsto q^{14}$ and $a = q$ to get that

$$1+\sum_{n=1}^{\infty}\frac{2q^n+q^{5n}-q^{9n}-2q^{13n}}{1-q^{14n}} = 1+2\sum_{n=0}^{\infty}\frac{q^{14n+1}}{1-q^{14n+1}}-2\sum_{n=1}^{\infty}\frac{q^{14n-1}}{1-q^{14n-1}}-\sum_{n=0}^{\infty}\frac{q^{14n+9}}{1-q^{14n+9}}+\sum_{n=1}^{\infty}\frac{q^{14n-9}}{1-q^{14n-9}}$$
$$= \frac{\left(q^{14};q^{14}\right)_\infty^2\left[q^2,q^8,q^8;q^{14}\right]_\infty}{\left[q,q,q^7,q^9;q^{14}\right]_\infty}.$$

Thus

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{2q^n+q^{5n}-q^{9n}-2q^{13n}}{1-q^{14n}}\right) = \frac{\left(q;q^2\right)_\infty\left(q^{14};q^{14}\right)_\infty^2\left[q^2,q^8,q^8;q^{14}\right]_\infty}{\left(q^2;q^2\right)_\infty\left[q,q,q^7,q^9;q^{14}\right]_\infty}$$
$$= \frac{\left(q^{14};q^{14}\right)_\infty\left[q^3,q^6;q^{14}\right]_\infty}{\left(q^7;q^{14}\right)_\infty\left[q,q^4;q^{14}\right]_\infty}.$$

To verify the second identity of (5.13), we divide both sides by $\frac{\left(q^{14};q^{14}\right)_\infty\left[q^3,q^6;q^{14}\right]_\infty}{\left(q^7;q^{14}\right)_\infty\left[q,q^4;q^{14}\right]_\infty}$, to get an identity between modular functions on $\Gamma_1(98)$. As we did in the proof of (5.4), we examine the orders at the poles of various modular functions and find that to prove the identity between modular functions we just need to verify the identity in the q-series expansion past q^{147}. We do this in Maple.

Next we claim that

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(\sum_{n=1}^{\infty}\frac{\left(-q^n+q^{3n}+q^{5n}-q^{9n}-q^{11n}+q^{13n}\right)}{1-q^{14n}}\right) = -q\frac{\left(q^{14};q^{14}\right)_\infty}{\left(q^7;q^{14}\right)_\infty}. \tag{5.14}$$

This is actually Entry 17(i) of [9], so there is nothing for us to prove. Lastly we claim that

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(\sum_{n=1}^{\infty}\frac{\left(-q^{3n}+2q^{5n}-2q^{9n}+q^{11n}\right)}{1-q^{14n}}\right)$$
$$= -q^3\frac{\left(q^{14};q^{14}\right)_\infty\left[q,q^2;q^{14}\right]_\infty}{\left(q^7;q^{14}\right)_\infty\left[q^5,q^6;q^{14}\right]_\infty}$$
$$= -q^8\frac{\left(q^{98};q^{98}\right)_\infty\left[q^{21};q^{98}\right]_\infty}{\left(q^{49};q^{98}\right)_\infty\left[q^{28};q^{98}\right]_\infty}-q^3\frac{\left(q^{49};q^{49}\right)_\infty\left[q^7;q^{49}\right]_\infty}{\left[q^{21};q^{49}\right]_\infty\left[q^{14};q^{98}\right]_\infty}+q^4\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{28};q^{98}\right]_\infty}+q^5\frac{\left(q^{49};q^{49}\right)_\infty\left[q^{21};q^{49}\right]_\infty}{\left[q^{14};q^{49}\right]_\infty\left[q^{42};q^{98}\right]_\infty}$$
$$-q^6\frac{\left(q^{49};q^{49}\right)_\infty}{\left[q^{42};q^{98}\right]_\infty}. \tag{5.15}$$

For this we apply (5.9) with $q \mapsto q^7$ and $a = b = q^2$ to get that

$$\sum_{n=1}^{\infty}\frac{\left(q^{3n}-2q^{5n}+2q^{9n}-q^{11n}\right)}{1-q^{14n}} = q^3\frac{\left(q^{14};q^{14}\right)_\infty^4\left[q^2,q^2,q^4;q^{14}\right]_\infty}{\left(q^7;q^7\right)_\infty^2\left[q^9,q^9,q^{11};q^{14}\right]_\infty}.$$

Thus

$$\frac{\left(q;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\left(\sum_{n=1}^{\infty}\frac{\left(-q^{3n}+2q^{5n}-2q^{9n}+q^{11n}\right)}{1-q^{14n}}\right) = -q^3\frac{\left(q;q^2\right)_\infty\left(q^{14};q^{14}\right)_\infty^4\left[q^2,q^2,q^4;q^{14}\right]_\infty}{\left(q^2;q^2\right)_\infty\left(q^7;q^7\right)_\infty^2\left[q^9,q^9,q^{11};q^{14}\right]_\infty}$$
$$= -q^3\frac{\left(q^{14};q^{14}\right)_\infty\left[q,q^2;q^{14}\right]_\infty}{\left(q^7;q^{14}\right)_\infty\left[q^5,q^6;q^{14}\right]_\infty}.$$

To verify the second identity of (5.15), we divide both sides by $q^3\frac{\left(q^{14};q^{14}\right)_\infty\left[q,q^2;q^{14}\right]_\infty}{\left(q^7;q^{14}\right)_\infty\left[q^5,q^6;q^{14}\right]_\infty}$, to get an identity between modular functions on $\Gamma_1(98)$. We examine the orders at the poles of various modular functions and find that to prove the identity between modular functions we just need to verify the identity in the q-series expansion past q^{147}. We do this in Maple.

Equation (5.8) now follows from (5.13), (5.14), and (5.15).

□

6. Dissections for $S_{G4}(z,q)$ and $S_{AG4}(z,q)$

Here we find the 5-dissections of $S_{G4}(\zeta_5,q)$ and $S_{AG4}(\zeta_5,q)$ using the techniques in [18]. Atkin and Swinnerton-Dyer pioneered this method to study the rank of partitions [7]. Since then Lovejoy and Osburn used it to study the Dyson rank of overpartitions [20], the M_2-rank of overpartitions [22], and the M_2-rank of partitions without repeated odd parts [21]. Also Ekin demonstrated that it could be used for the crank of partitions [12]. However, it is quicker to derive these dissections from the product and series forms of $S_{G4}(z,q)$ and $S_{AG4}(z,q)$. For this reason, we omit some of the details but do include the general identities that lead to the end results. To begin we use Bailey's Lemma with $\rho_1 = z$ and $\rho_2 = z^{-1}$ to get that

$$\begin{aligned}
S_{G4}(z,q) &= \frac{\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=0}^{\infty}\left(z,z^{-1};q^2\right)_n q^{2n}\beta_n^{G4} - \frac{\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\\
&= \frac{\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty}{(1-z)(1-z^{-1})\left(q^2;q^2\right)_\infty^2}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n^2+3n}{2}}(1+q^n)}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right)\\
&\quad - \frac{\left(q^4;q^4\right)_\infty\left(-q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\\
&= \frac{1}{(1-z)(1-z^{-1})\left(q;q\right)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n^2+3n}{2}}(1+q^n)}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) - \frac{\left(q^2;q^2\right)_\infty}{\left(q,z,z^{-1};q^2\right)_\infty}.
\end{aligned}$$

Similarly, for $S_{AG4}(z,q)$, Bailey's Lemma gives that

$$S_{AG4}(z,q) = \frac{1}{(1-z)(1-z^{-1})\left(q;q\right)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n^2+n}{2}}(1+q^{3n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) - \frac{\left(q^2;q^2\right)_\infty}{\left(q,z,z^{-1};q^2\right)_\infty}.$$

Proposition 6.1.

$$\begin{aligned}
\frac{\left(q^2;q^2\right)_\infty}{\left(q,\zeta_5q^2,\zeta_5^{-1}q^2;q^2\right)_\infty} &= \frac{\left(q^{25};q^{25}\right)_\infty\left[q^{20};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty\left[q^{10};q^{50}\right]_\infty} + (\zeta_5+\zeta_5^4)q^5\frac{\left(q^{50};q^{50}\right)_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{20};q^{50}\right]_\infty} + q\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{5};q^{25}\right]_\infty}\\
&\quad + (\zeta_5+\zeta_5^4)q^2\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{25}\right]_\infty} + (\zeta_5+\zeta_5^4)q^3\frac{\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty\left[q^{20};q^{50}\right]_\infty}\\
&\quad + q^3\frac{\left(q^{50};q^{50}\right)_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{10};q^{50}\right]_\infty}.
\end{aligned}$$

Proof. By Gauss and the Jacobi Triple Product Identity we have

$$\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty} = \frac{1}{2}\sum_{n=-\infty}^{\infty}q^{n(n+1)/2} = \frac{\left(q^{25};q^{25}\right)_\infty\left[q^{20};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty} + q\frac{\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty} + q^3\frac{\left(q^{50};q^{50}\right)_\infty}{\left(q^{25};q^{50}\right)_\infty}.$$

By Lemma 3.9 of [16], with q replaced by q^2 we have

$$\frac{1}{\left(\zeta_5q^2,\zeta_5^{-1}q^2;q^2\right)_\infty} = \frac{1}{\left[q^{10};q^{50}\right]_\infty} + (\zeta_5+\zeta_5^4)\frac{q^2}{\left[q^{20};q^{50}\right]_\infty}.$$

Multiplying these two identities together gives the result. ∎

With Proposition 6.1 we find (2.12) and (2.13) to be equivalent to the following.

Proposition 6.2.

$$\begin{aligned}
&\frac{1}{\left(q;q\right)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^n q^{\frac{n^2+3n}{2}}(1+q^n)}{(1-\zeta_5q^{2n})(1-\zeta_5^{-1}q^{2n})}\right)\\
&= \frac{\left(q^{25};q^{25}\right)_\infty\left[q^{20};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty\left[q^{10};q^{50}\right]_\infty} + (1-2\zeta_5-2\zeta_5^4)q^5\frac{\left(q^{50};q^{50}\right)_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{20};q^{50}\right]_\infty}\\
&\quad - (1+2\zeta_5+2\zeta_5^4)q^{10}\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{10};q^{200}\right]_\infty}{\left[q^{10};q^{50}\right]_\infty\left[q^{5};q^{100}\right]_\infty} + \frac{q\left(q^{25};q^{25}\right)_\infty}{\left[q^{5};q^{25}\right]_\infty}
\end{aligned}$$

$$- (2-\zeta_5-\zeta_5^4)q^6\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{30};q^{200}\right]_\infty}{\left[q^{10};q^{50}\right]_\infty\left[q^{15};q^{100}\right]_\infty} + (\zeta_5+\zeta_5^4)\frac{q^2\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{25}\right]_\infty}$$
$$- (2-\zeta_5-\zeta_5^4)q^{12}\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{10};q^{200}\right]_\infty}{\left[q^{20};q^{50}\right]_\infty\left[q^{5};q^{100}\right]_\infty} + (-1+\zeta_5+\zeta_5^4)q^3\frac{\left(q^{50};q^{50}\right)_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{10};q^{50}\right]_\infty}$$
$$+ (\zeta_5+\zeta_5^4)\frac{q^3\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty\left[q^{20};q^{50}\right]_\infty} + (1-3\zeta_5-3\zeta_5^4)q^8\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{30};q^{200}\right]_\infty}{\left[q^{20};q^{50}\right]_\infty\left[q^{15};q^{100}\right]_\infty},$$
$$\frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^n q^{\frac{n^2+n}{2}}(1+q^{3n})}{(1-\zeta_5 q^{2n})(1-\zeta_5^{-1}q^{2n})}\right)$$
$$= \frac{\left(q^{25};q^{25}\right)_\infty\left[q^{20};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty\left[q^{10};q^{50}\right]_\infty} + (\zeta_5+\zeta_5^4)q^5\frac{\left(q^{50};q^{50}\right)_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{20};q^{50}\right]_\infty} + (\zeta_5+\zeta_5^4-2)q^{10}\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{10};q^{200}\right]_\infty}{\left[q^{10};q^{50}\right]_\infty\left[q^{5};q^{100}\right]_\infty}$$
$$+ (\zeta_5+\zeta_5^4-2)q\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{70};q^{200}\right]_\infty}{\left[q^{10};q^{50}\right]_\infty\left[q^{35};q^{100}\right]_\infty} + q\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{5};q^{25}\right]_\infty}$$
$$- (1+2\zeta_5+2\zeta_5^4)q^6\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{30};q^{200}\right]_\infty}{\left[q^{10};q^{50}\right]_\infty\left[q^{15};q^{100}\right]_\infty} + (\zeta_5+\zeta_5^4)q^2\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{25}\right]_\infty}$$
$$+ (1-3\zeta_5-3\zeta_5^4)q^{12}\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{10};q^{200}\right]_\infty}{\left[q^{20};q^{50}\right]_\infty\left[q^{5};q^{100}\right]_\infty} + (1-3\zeta_5-3\zeta_5^4)q^3\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{70};q^{200}\right]_\infty}{\left[q^{20};q^{50}\right]_\infty\left[q^{35};q^{100}\right]_\infty}$$
$$+ (\zeta_5+\zeta_5^4)q^3\frac{\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty\left[q^{20};q^{50}\right]_\infty} + q^3\frac{\left(q^{50};q^{50}\right)_\infty}{\left(q^{25};q^{50}\right)_\infty\left[q^{10};q^{50}\right]_\infty}$$
$$+ (-2+\zeta_5+\zeta_5^4)q^8\frac{\left(q^{100};q^{100}\right)_\infty\left[q^{30};q^{200}\right]_\infty}{\left[q^{20};q^{50}\right]_\infty\left[q^{15};q^{100}\right]_\infty}.$$

The major work in proving identities like this is to find identities that allow us to see the series terms on the left hand sides as products. To this end, we note that

$$\begin{aligned}
&1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n^2+3n}{2}}(1+q^n)}{(1-zq^{2n})(1-z^{-1}q^{2n})}\\
&=\sum_{n=-\infty}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{\frac{n^2+3n}{2}}}{(1-zq^{2n})(1-z^{-1}q^{2n})}\\
&=\sum_{n=-\infty}^{\infty}\frac{(1-z)(1-z^{-1})q^{2n^2+3n}}{(1-zq^{4n})(1-z^{-1}q^{4n})}-\sum_{n=-\infty}^{\infty}\frac{(1-z)(1-z^{-1})q^{2n^2+5n+2}}{(1-zq^{4n+2})(1-z^{-1}q^{4n+2})}.
\end{aligned}$$

We then define

$$V_\ell(b)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{2n^2+bn}}{1-q^{4\ell n}},\qquad U_\ell(b)=\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+bn}}{1-q^{4\ell n+2\ell}}.$$

We note replacing n by $-n$ gives that

$$V_\ell(b)=-V_\ell(4\ell-b),\qquad U_\ell(b)=-q^{2\ell-b+2}U_\ell(4\ell+4-b).$$

Next we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{(1-\zeta_5)(1-\zeta_5^{-1})q^{2n^2+3n}}{(1-\zeta_5 q^{4n})(1-\zeta_5^{-1}q^{4n})}\\
&=1+\sum_{n\neq 0}\frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^n q^{2n^2+3n}(1-q^{4n})(1-\zeta_5^2q^{4n})(1-\zeta_5^3q^{4n})}{(1-q^{20n})}\\
&=1+(2-\zeta_5-\zeta_5^4)(V_5(3)-V_5(15))+(-1+3\zeta_5+3\zeta_5^4)(V_5(7)-V_5(11))\\
&=1+(2-\zeta_5-\zeta_5^4)(V_5(3)+V_5(5))+(-1+3\zeta_5+3\zeta_5^4)(V_5(7)+V_5(9)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{(1-\zeta_5)(1-\zeta_5^{-1})q^{2n^2+5n+2}}{(1-\zeta_5 q^{4n+2})(1-\zeta_5^{-1}q^{4n+2})}\\
&= \sum_{n\neq-\infty}^{\infty} \frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^n q^{2n^2+5n+2}(1-q^{4n+2})(1-\zeta_5^2 q^{4n+2})(1-\zeta_5^3 q^{4n+2})}{(1-q^{20n+10})}\\
&= (2-\zeta_5-\zeta_5^4)(q^2U_5(5)-q^8U_5(17)) + (-1+3\zeta_5+3\zeta_5^4)(q^4U_5(9)-q^6U_5(13))\\
&= (2-\zeta_5-\zeta_5^4)(q^2U_5(5)+q^3U_5(7)) + (-1+3\zeta_5+3\zeta_5^4)(q^4U_5(9)+q^5U_5(11)).
\end{aligned}$$

Thus

$$\begin{aligned}
&1+\sum_{n=1}^{\infty} \frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^n q^{\frac{n^2+3n}{2}}(1+q^n)}{(1-\zeta_5 q^{2n})(1-\zeta_5^{-1}q^{2n})}\\
&= 1+(2-\zeta_5-\zeta_5^4)(V_5(3)-q^2U_5(5)+V_5(5)-q^3U_5(7))\\
&\quad + (-1+3\zeta_5+3\zeta_5^4)(V_5(7)-q^4U_5(9)+V_5(9)-q^5U_5(11)).
\end{aligned}$$

In the same fashion we deduce that

$$\begin{aligned}
&1+\sum_{n=1}^{\infty} \frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^n q^{\frac{n^2+n}{2}}(1+q^{3n})}{(1-\zeta_5 q^{2n})(1-\zeta_5^{-1}q^{2n})}\\
&= 1+(2-\zeta_5-\zeta_5^4)(V_5(1)-qU_5(3)+V_5(7)-q^4U_5(9))\\
&\quad + (-1+3\zeta_5+3\zeta_5^4)(V_5(5)-q^3U_5(7)-V_5(9)+q^5U_5(11)).
\end{aligned}$$

Similar to Ekin's work in [12], we use the functions

$$\begin{aligned}
T(z,w,q) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} w^n}{1-zq^n},\\
T^*(w,q) &= \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(n+1)/2} w^n}{1-q^n},\\
h(z,q) &= T^*(z^{-1},q) + zT(z^2,z,q).
\end{aligned}$$

Lemma 6.3. *For b and ℓ odd integers with $\ell > 1$, we have*

$$\begin{aligned}
&V_\ell(b) - q^{\frac{b+1}{2}} U_\ell(b+2)\\
&= h(-q^{2\ell^2-b\ell}, q^{4\ell^2}) + \left(q^{4\ell^2};q^{4\ell^2}\right)_\infty^2 \left[-q^{2\ell^2-b\ell};q^{4\ell^2}\right]_\infty \sum_{k=1}^{\ell-1} q^{2k^2+bk} \frac{\left[q^{2b\ell+8k\ell};q^{4\ell^2}\right]_\infty}{\left[q^{4k\ell}, q^{2b\ell+4k\ell}, -q^{2\ell^2+b\ell+4k\ell};q^{4\ell^2}\right]_\infty}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&V_\ell(b) - q^{\frac{b+1}{2}} U_\ell(b+2)\\
&= \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{q^{2n^2+bn}}{1-q^{4\ell n}} - q^{\frac{b+1}{2}} \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+bn+2n}}{1-q^{4\ell n+2\ell}}\\
&= \sum_{k=0}^{\ell-1} q^{2k^2+bk} \sum_{\substack{n=-\infty\\ (n,k)\neq(0,0)}}^{\infty} \frac{q^{2\ell^2n^2+b\ell n+4k\ell n}}{1-q^{4\ell^2 n+4k\ell}} - \sum_{k=0}^{\ell-1} q^{2k^2+bk-4k\ell-b\ell+2\ell^2} \sum_{n=-\infty}^{\infty} \frac{q^{2\ell^2n^2+4\ell^2 n-4k\ell n-b\ell n}}{1-q^{4\ell^2 n+4\ell^2-2b\ell-4k\ell}}\\
&= \sum_{k=0}^{\ell-1} q^{2k^2+bk} \sum_{\substack{n=-\infty\\ (n,k)\neq(0,0)}}^{\infty} \frac{(-1)^n q^{2\ell^2 n(n+1)}(-q)^{b\ell n+4k\ell n-2\ell^2 n}}{1-q^{4\ell^2 n+4k\ell}}
\end{aligned}$$

$$
\begin{aligned}
&\quad -\sum_{k=0}^{\ell-1} q^{2k^2+bk-4k\ell-b\ell+2\ell^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2\ell^2 n(n+1)}(-q)^{2\ell^2 n-4k\ell n-b\ell n}}{1-q^{4\ell^2 n+4\ell^2-2b\ell-4k\ell}} \\
&= T^*(-q^{b\ell-2\ell^2}, q^{4\ell^2}) - q^{2\ell^2-b\ell} T(q^{4\ell^2-2b\ell}, -q^{2\ell^2-b\ell}, q^{4\ell^2}) \\
&\quad + \sum_{k=1}^{\ell-1} q^{2k^2+bk} \left(T(q^{4k\ell}, -q^{b\ell+4k\ell-2\ell^2}, q^{4\ell^2}) - q^{-4k\ell-b\ell+2\ell^2} T(q^{4\ell^2-2b\ell-4k\ell}, -q^{2\ell^2-4k\ell-b\ell}, q^{4\ell^2}) \right) \\
&= h(-q^{2\ell^2-b\ell}, q^{4\ell^2}) \\
&\quad + \sum_{k=1}^{\ell-1} q^{2k^2+bk} \left(T(q^{4k\ell}, -q^{b\ell+4k\ell-2\ell^2}, q^{4\ell^2}) - q^{-4k\ell-b\ell+2\ell^2} T(q^{4\ell^2-2b\ell-4k\ell}, -q^{2\ell^2-4k\ell-b\ell}, q^{4\ell^2}) \right).
\end{aligned}
$$

Here we have replaced n by $\ell n + k$ in $V_\ell(b)$ and replaced n by $\ell n - k + \ell - \frac{b+1}{2}$ in $U_\ell(b+2)$. By Lemma 2 of [19] we have that

$$
wT(zw, w, q) + T(zw^{-1}, w^{-1}, q) = \frac{(q;q)_\infty^2 \left[z, w^2; q\right]_\infty}{\left[zw^{-1}, zw, w; q\right]_\infty}.
$$

We note one could also deduce this identity using Theorem 2.1 of [11]. Applying this with $q \mapsto q^{4\ell^2}$, $w = -q^{2\ell^2-b\ell-4k\ell}$, $z = -q^{2\ell^2-b\ell}$ yields

$$
\begin{aligned}
&T(q^{4k\ell}, -q^{b\ell+4k\ell-2\ell^2}, q^{4\ell^2}) - q^{-4k\ell-b\ell+2\ell^2} T(q^{4\ell^2-2b\ell-4k\ell}, -q^{2\ell^2-4k\ell-b\ell}, q^{4\ell^2}) \\
&= \frac{\left(q^{4\ell^2}; q^{4\ell^2}\right)_\infty^2 \left[-q^{2\ell^2-b\ell}, q^{4\ell^2-2b\ell-8k\ell}; q^{4\ell^2}\right]_\infty}{\left[q^{4k\ell}, q^{4\ell^2-2b\ell-4k\ell}, -q^{2\ell^2-b\ell-4k\ell}; q^{4\ell^2}\right]_\infty} \\
&= \frac{\left(q^{4\ell^2}; q^{4\ell^2}\right)_\infty^2 \left[-q^{2\ell^2-b\ell}, q^{2b\ell+8k\ell}; q^{4\ell^2}\right]_\infty}{\left[q^{4k\ell}, q^{2b\ell+4k\ell}, -q^{2\ell^2+b\ell+4k\ell}; q^{4\ell^2}\right]_\infty}.
\end{aligned}
$$

This completes the proof. ◻

Also one can express combinations of $h(z,q)$ in terms of products by using Lemma 1 of [12]. For our purposes, we could use the following, the proof of which is basic algebra and applying Lemma 1 of [12].

Proposition 6.4.

$$
\begin{aligned}
&(3a-d)h(-q^5, q^{100}) + (3b-a)h(-q^{15}, q^{100}) + (3c-b)h(-q^{45}, q^{100}) + (3d-c)h(-q^{35}, q^{100}) + c + d \\
&= a\frac{\left(q^{100};q^{100}\right)_\infty^2 \left[q^{10};q^{100}\right]_\infty^3}{\left[-q^5;q^{100}\right]_\infty^3 \left[-q^{15};q^{100}\right]_\infty} + (c-a)\frac{\left(q^{100};q^{100}\right)_\infty^2 \left[q^{20};q^{100}\right]_\infty^3}{\left[q^{10};q^{100}\right]_\infty^3 \left[q^{30};q^{100}\right]_\infty} + b\frac{\left(q^{100};q^{100}\right)_\infty^2 \left[q^{30};q^{100}\right]_\infty^3}{\left[-q^{15};q^{100}\right]_\infty^3 \left[-q^{45};q^{100}\right]_\infty} \\
&\quad + (d-b)\frac{\left(q^{100};q^{100}\right)_\infty^2 \left[q^{40};q^{100}\right]_\infty^3}{\left[q^{30};q^{100}\right]_\infty^3 \left[q^{10};q^{100}\right]_\infty} + cq^{35}\frac{\left(q^{100};q^{100}\right)_\infty^2 \left[q^{10};q^{100}\right]_\infty^3}{\left[-q^{45};q^{100}\right]_\infty^3 \left[-q^{35};q^{100}\right]_\infty} + dq^5\frac{\left(q^{100};q^{100}\right)_\infty^2 \left[q^{30};q^{100}\right]_\infty^3}{\left[-q^{35};q^{100}\right]_\infty^3 \left[-q^5;q^{100}\right]_\infty}.
\end{aligned}
$$

Also we note that

$$
h(-q^{25}, q^{100}) = \frac{\left(q^{100};q^{100}\right)_\infty^2 \left[q^{50};q^{100}\right]_\infty^4}{\left[-q^{25};q^{100}\right]_\infty^4} = \frac{\left(q^{25};q^{25}\right)_\infty^4}{\left(q^{100};q^{100}\right)_\infty^2}.
$$

This would allow use to express the identities in Proposition 6.2 just in terms of infinite products. We could then rewrite the identities strictly in terms of modular functions. The identity in terms of modular functions could then be proved as we did for the identities in (5.4), (5.13), and (5.15). We do not include these calculations here.

7. Proof of Corollary 2.4

We note that Theorem 2.3 immediately gives that the coefficients of q^{3n} in $S_{F3}(\zeta_3,q)$, q^{5n+1} in $S_{B2}(\zeta_5,q)$, q^{5n+4} in $S_{B2}(\zeta_5,q)$, q^{5n} in $S_{F3}(\zeta_5,q)$, q^{5n+4} in $S_{F3}(\zeta_5,q)$, q^{5n+4} in $S_{G4}(\zeta_5,q)$, q^{5n+4} in $S_{AG4}(\zeta_5,q)$, q^{7n+1} in $S_{B2}(\zeta_7,q)$, q^{7n+5} in $S_{B2}(\zeta_7,q)$, q^{7n} in $S_{F3}(\zeta_7,q)$, q^{7n+4} in $S_{F3}(\zeta_7,q)$, and q^{7n+6} in $S_{F3}(\zeta_7,q)$ are all zero. Thus the identities in Corollary 2.4 for $B2$, $F3$, $G4$, and $AG4$ follow. We still need to prove that the coefficients of q^{3n+2} in $S_{J1}(\zeta_3,q)$, q^{3n} in $S_{J2}(\zeta_3,q)$, and q^{3n+1} in $S_{J3}(\zeta_3,q)$ are zero. We prove this by using Theorem 2.2.

For $J1$, we use (2.1) to see that

$$\begin{aligned}&(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^{-1})\left(q^3;q^3\right)_\infty S_{J1}(\zeta_3,q)\\ &=\sum_{j=2}^{\infty}\frac{(1-\zeta_3^j)(1-\zeta_3^{j-1})\zeta_3^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}(1-q^j-q^{2j-2}+q^{4j-3}+q^{5j-2}-q^{6j-3})}{(1-q^{3j-3})(1-q^{3j})}.\end{aligned}$$

Thus the non-zero terms occur only when $j\equiv 2 \pmod 3$, but one finds that $q^{\frac{j(j-1)}{2}}(1-q^j-q^{2j-2}+q^{4j-3}+q^{5j-2}-q^{6j-3})$ only contributes terms of the form q^{3n} and q^{3n+1} when $j\equiv 2 \pmod 3$. Thus $S_{J1}(\zeta_3,q)$ has no non-zero terms of the form q^{3n+2}.

For $J2$, we use (2.2) to see that

$$\begin{aligned}&(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^{-1})\left(q^3;q^3\right)_\infty S_{J2}(\zeta_3,q)\\ &=\sum_{j=2}^{\infty}\frac{(1-\zeta_3^j)(1-\zeta_3^{j-1})\zeta_3^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}(1-q^{j-1}-q^{2j}+q^{4j-1}+q^{5j-3}-q^{6j-3})}{(1-q^{3j-3})(1-q^{3j})}.\end{aligned}$$

Thus the non-zero terms occur only when $j\equiv 2 \pmod 3$, but one finds that $q^{\frac{j(j-1)}{2}}(1-q^{j-1}-q^j-q^{2j}+q^{4j-1}+q^{5j-3}-q^{6j-3})$ only contributes terms of the form q^{3n+1} and q^{3n+2} when $j\equiv 2 \pmod 3$. Thus $S_{J2}(\zeta_3,q)$ has no non-zero terms of the form q^{3n}.

For $J3$, we use (2.3) to see that

$$\begin{aligned}&(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^{-1})\left(q^3;q^3\right)_\infty S_{J3}(\zeta_3,q)\\ &=\sum_{j=2}^{\infty}\frac{(1-\zeta_3^j)(1-\zeta_3^{j-1})\zeta_3^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}(q^{j-1}-q^j-q^{2j-2}+q^{2j}+q^{4j-3}-q^{4j-1}-q^{5j-3}+q^{5j-2})}{(1-q^{3j-3})(1-q^{3j})}.\end{aligned}$$

Thus the non-zero terms occur only when $j\equiv 2 \pmod 3$, but one finds that $q^{\frac{j(j-1)}{2}}(q^{j-1}-q^j-q^{2j-2}+q^{2j}+q^{4j-3}-q^{4j-1}-q^{5j-3}+q^{5j-2})$ only contributes terms of the form q^{3n} and q^{3n+1} when $j\equiv 2 \pmod 3$. Thus $S_{J3}(\zeta_3,q)$ has no non-zero terms of the form q^{3n+1}.

8. Concluding Remarks

We could also prove dissection identities for $S_{J2}(\zeta_3,q)$ and $S_{J3}(\zeta_3,q)$ in the same way we proved the dissections for $S_{G4}(\zeta_5,q)$ and $S_{AG4}(\zeta_5,q)$. It would require defining functions similar to V_ℓ, U_ℓ, T, and h and finding the appropriate identities. Whereas $S_{G4}(z,q)$ and $S_{AG4}(z,q)$ use functions and formulas similar to those used for the crank, $S_{J2}(z,q)$ and $S_{J3}(z,q)$ would use functions and formulas similar to those used for the rank. We save this for another time.

We might think to try the methods of this paper with the following Bailey pairs relative to $(1,q^2)$,

$$\begin{aligned}
\beta_n^{F1}&=\frac{1}{(q,q^2;q^2)_n}, & \alpha_n^{F1}&=\begin{cases}1 & \text{if } n=0\\ q^{2n^2-n}(1+q^{2n}) & \text{if } n\ge 1\end{cases},\\
\beta_n^{G*}&=\frac{q^{n^2-2n}}{(q^4;q^4)_n\,(q;q^2)_n}, & \alpha_n^{G*}&=\begin{cases}1 & \text{if } n=0\\ (-1)^{n(n-1)/2}q^{n(n-3)/2}(1+(-1)^nq^{3n}) & \text{if } n\ge 1\end{cases}\\
\beta_n^{G**}&=\frac{q^{n^2}}{(q^4;q^4)_n\,(q;q^2)_n}, & \alpha_n^{G**}&=\begin{cases}1 & \text{if } n=0\\ (-1)^{n(n+1)/2}q^{n(n-1)/2}(1+(-1)^nq^{n}) & \text{if } n\ge 1\end{cases}.
\end{aligned}$$

These Bailey pairs are $F(1)$ and the first two Bailey pairs a listed on page 470 of [26]. We would then define the following series.

$$S_{F1}(z,q) = \frac{\left(q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=1}^{\infty}\left(z,z^{-1};q^2\right)_n q^{2n}\beta_n^{F1} = \frac{\left(q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n q^{2n}}{\left(q,q^2;q^2\right)_n},$$

$$S_{G*}(z,q) = \frac{\left(q^4;q^4\right)_\infty\left(q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=1}^{\infty}\left(z,z^{-1};q^2\right)_n q^{2n}\beta_n^{G*} = \frac{\left(q^4;q^4\right)_\infty\left(q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n q^{n^2}}{\left(q^4;q^4\right)_n\left(q;q^2\right)_n},$$

$$S_{G**}(z,q) = \frac{\left(q^4;q^4\right)_\infty\left(q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=1}^{\infty}\left(z,z^{-1};q^2\right)_n q^{2n}\beta_n^{G*} = \frac{\left(q^4;q^4\right)_\infty\left(q;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n q^{n^2+2n}}{\left(q^4;q^4\right)_n\left(q;q^2\right)_n}.$$

At first it appears that these series may explain congruences for other new spt functions, however, they are old functions in disguise. In particular one finds that $S_{G*}(z,-q) = S_{AG4}(z,q)$ and $S_{G**}(z,-q) = S_{G4}(z,q)$. Similarly $S_{F1}(z,-q) = S2(z,q)$, where $S2(z,q)$ is a two variable generalization for the M2spt function studied in [14].

While this paper gives the last of the spt-crank-type functions for Bailey pairs from [26] and [27], with such simple linear congruences, we should expect there to be many more interesting spt-crank-type functions. There are plenty of other Bailey pairs from other sources that may lead to new functions. Also we have not used all the Bailey pairs from [26] and [27], we have only used those that have simple congruences. So far all Bailey pairs have been relative to (a,q) with $a=1$, but a slight change in the form of the spt-crank-type functions may allow for many useful functions coming from other values of a. In a coming paper, we investigate Bailey pairs arising from variations of Bailey's Lemma and conjugate Bailey pairs.

The functions studied here and in [18] and [15] may have additional properties worth studying. While the $M_{A1}(m,n)$ were given a combinatorial interpretation in [18] (in particular they are non-negative), work on the other $M_X(m,n)$ still needs to be done. Additionally, the original spt functions for partitions and overpartitions are known to be related to mock modular forms and harmonic Maass forms. Any of the other spt functions that can be expressed in terms of known rank functions and infinite products will also lead to harmonic Maass forms, so these functions can be studied from that aspect as well.

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